ITERATIVE ALGORITHMS FOR FINDING A COMMON FIXED POINT OF GENERALIZED NONEXPANSIVE SEQUENCES IN A HILBERT SPACE

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ชื่อเรื่อง : ขั้นตอนวิธีทำซ้ำสำหรับการหาจุดตรึงร่วมของลำดับการส่งแบบไม่ขยายที่วางนัยทั่วไป ในปริภูมิฮิลเบิร์ต

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วิทยานิพนธ์นี้ได้เสนอขั้นตอนวิธีทำซ้ำทั่วไปใหม่สำหรับคู่ของลำดับของการส่งแบบกึ่งไม่ขยายที่สอดคล้องกับเงื่อนไขบางประการ และนำเสนอทฤษฎีการลู่เข้าแบบเข้มสู่จุดตรึงร่วมของลำดับการส่งนี้ สำหรับขั้นตอนวิธีทำซ้ำดังกล่าวในปริภูมิฮิลเบิร์ต ภายใต้สมมติฐานที่เหมาะสมสามารถอธิบายผลลัพธ์ของงานวิจัยที่เกี่ยวข้องได้ อาทิเช่น Aoyama และ Kimura และ Tian และ Jin, และ Wongchan และ Saejung ในตอนท้ายได้ประยุกต์เพื่อหาสมาชิกร่วมของเซตของจุดตรึงร่วมของลำดับการส่งแบบกึ่งไม่ขยาย เซตคำตอบของปัญหาดุลยภาพ และเซตคำตอบของการหาศูนย์สำหรับผลบวกของสองการส่งทางเดียวอีกด้วย

ABSTRACT

TITLE	: ITERATIVE ALGORITHMS FOR FINDING A COMMON FIXED POINT OF GENERALIZED NONEXPANSIVE SEQUENCES IN A HILBERT SPACE
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In this research, we introduce a new general iterative method for a pair of sequences of quasi-nonexpansive mappings satisfying certain conditions and present strong convergence theorems which the iteration converge to a common fixed point of these mappings in a Hilbert space. With an appropriate setting, we obtain the corresponding results due to Aoyama and Kimura, Tian and Jin, and Wongchan and Saejung. At the end, we also apply our methods to find a common element of the set of fixed points for a sequence of quasi-nonexpansive mappings, the set of solutions of an equilibrium problem and the set of zero points for the sum of two monotone mappings.

CONTENTS

CHAPTER I

INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ be a mapping. The set of all fixed points of T is called the *fixed point* of T and denoted by $F(T)$. A mapping T is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$\|x - Ty\| \leq \|x - y\|, \quad \forall (x, y) \in F(T) \times C;$$

(iii) strongly nonexpansive [7] if it is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (Tx_n - Ty_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\lim_{n\to\infty} \big(\|x_n - y_n\| - \|Tx_n - Ty_n\|\big) = 0;$$

(iv) strongly quasi-nonexpansive [22] if it is quasi-nonexpansive and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

whenever $\{x_n\}$ is a bounded sequence in C such that

$$\lim_{n\to\infty} \big(\|x_n - z\| - \|Tx_n - z\|\big) = 0$$

for some $z \in F(T)$.

It follows directly from the above definitions that

(i) if T is nonexpansive with a nonempty fixed point set, then T is quasi-nonexpansive;

(ii) if T is strongly nonexpansive with a nonempty fixed point set, then T is strongly quasi-nonexpansive.

The process for approximation of a fixed point of a nonexpansive or a quasi-nonexpansive mapping is one of interesting problems in mathematics and it has been investigated by many researchers. Wittmann [36] studied the following iteration scheme, which was first considered by Halpern [11]

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where a sequence $\{\alpha_n\}$ in $(0,1)$ is chosen such that

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

see also Reich [23] and Xu [38]. Wittmann proved that for any $x_1 = x \in C$, the sequence $\{x_n\}$ defined by (1.1) converges strongly to the unique element $P_{F(T)}x \in F(T)$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$. Moudafi [18] proposed the scheme which is known as Moudafi's viscosity approximation process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $f : C \to C$ is an α-contraction, $T : C \to C$ is nonexpansive with $F(T) \neq \varnothing$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying conditions (C1), (C2) and (C3). He proved that $\{x_n\}$ defined by (1.2) converges strongly to $z \in F(T)$ and the following inequality holds

$$\langle f(z) - z, q - z \rangle \leq 0, \quad \forall q \in F(T).$$

In the literature, Moudafi's scheme has been widely studied and extended by Cianciaruso et al. [8], Peng and Yao [20], Saejung [24], Suzuki [26] and references therein.

Recently, Wongchan and Saejung [37] improved and extended this result to obtain a strong convergence theorem for a strongly quasi-nonexpansive mapping T such that $I - T$ is demiclosed at zero under only the conditions (C1) and (C2).

Very recently, Aoyama and Kimura [2] presented a strong convergence theorem for a pair of sequences of nonexpansive mappings in a Hilbert space by the following iteration:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S_n\big(\alpha_n x + (1-\alpha_n) T_n x_n\big), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{S_n\}$ and $\{T_n\}$ are sequences of nonexpansive self mappings of C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. They proved that $\{x_n\}$ converges strongly to the nearest point of the set of common fixed points of $\{S_n\}$ and $\{T_n\}$ under some appropriate assumptions.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping T on H:

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.4}$$

where A is a certain operator on H and b is a given point in H. Marino and Xu [16] was combine the iterative method with the viscosity approximation method by

$$x_1 = x \in H, \quad x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where f is a contraction on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.6}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf(*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Tian and Jin [32, 33] studied the following iterative scheme:

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\big((1-\omega)I + \omega T\big)x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where f is a Lipschitzian continuous operator on H, $T : H \to H$ is quasi-nonexpansive, $\{\alpha_n\}$ is a sequence in (0,1), and $\omega \in (0,1)$. It is proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0, \forall x \in C,$ where $\tilde{x} = P_{F(T)}(I - A + \gamma f)\tilde{x}$.

Motivated by Aoyama and Kimura [2] and Tian and Jin [32, 33], we consider the following iterative process:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1 - \beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where A is a certain operator on H, $f : C \to H$ is a Lipschitzian continuous operator, $\{T_n : C \to H\}$ and $\{S_n : C \to H\}$ are sequences of quasi-nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1), and $\gamma > 0$. Strong convergence theorems of this scheme for finding a common fixed points of $\{S_n\}$ and $\{T_n\}$ are presented. With an appropriate setting, we shall obtain the corresponding results due to Aoyama and Kimura [2] and Tian and Jin [32, 33], and many others.

The thesis is organized as follows. Chapter II contains some preliminaries and basic concepts of normed space, inner product and Hibert space which are essential for the study. In Chapter III, we study the iteration scheme defined by (1.8) for a sequence of quasi-nonexpansive mappings and present the strong convergence theorems of this scheme to common fixed point of these mappings under some additional assumption on the mappings. Application of our main results for find a common element of the set of fixed points for a sequence of quasi-nonexpansive mappings, the set of solutions of an equilibrium problem and the set of zero points for the sum of two monotone mappings is showed. For the last chapter, we conclude our results of the studies.

CHAPTER II
PRELIMINARIES

In this chapter, we give some definitions, notations and theorems that will be used in the later chapter.

Throughout this study, we let $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real numbers and the set of natural numbers, respectively.

2.1 Normed spaces

Definition 2.1.1 ([17]). Let X be a real linear space. A *norm* on X is a nonnegative real valued function on X such that the following conditions are satisfied by $x, y \in X$ and each scalar α:

(N1) $\|x\| = 0$ if and only if $x = 0$;

(N2) $\|\alpha x\| = |\alpha|\|x\|$;

(N3) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a *normed space*. If no any confusion arises, we usually write the norm space $(X, \|\cdot\|)$ simply as X.

Proposition 2.1.2 ([17]). A normed space X is a *metric space* with the *metric induced by the norm* on X defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$.

Definition 2.1.3 ([13, Definition 4.8-1]). A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in norm*) if for every $\varepsilon > 0$ there are $x \in X$ and $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon, \quad \forall n \geq N.$$

This is written $\lim_{n\to\infty} x_n = x$ or simply $x_n \to x$. The element x is called the *strong limit* of $\{x_n\}$, and we also say that $\{x_n\}$ *converges strongly to* x.

Definition 2.1.4 ([13, Definition 2.3-1]). A sequence $\{x_n\}$ in a normed space X is said to be *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon, \quad \forall m, n \geq N.$$

Remark 2.1.5. Every convergent sequence is Cauchy, but the converse is not true.

Example 2.1.6. Let $\mathbb{Q}$ be a set of all rational numbers and $\mathbb{R}\backslash\mathbb{Q}$ be a set of all irrational numbers. Let $\bar{x} \in \mathbb{R}\backslash\mathbb{Q}$ and $x_n \in \mathbb{Q}$ be in the interval $\left(\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in $\mathbb{Q}$ but not converges.

Definition 2.1.7. A normed space X is called *a Banach space* if every Cauchy sequence in X converges to an element of itself.

Definition 2.1.8 ([13, Definition 1.4-1]). A nonempty subset C of a normed space X is called *a bounded set* if its diameter

$$\operatorname{diam} C = \sup_{x,y \in C} \|x - y\|$$

is finite. A sequence $\{x_n\}$ in X is called *bounded* if the set $\{x_1, x_2, \ldots\}$ is bounded.

Remark 2.1.9. A sequence $\{x_n\}$ in X is bounded if and only if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.1.10 ([13]). *Let X be a normed space. Then the following hold.*

(i) *A convergent sequence in X is bounded and its limit is unique.*

(ii) *If a sequence $\{x_n\}$ in X is convergent to x, then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to same limit x.*

Theorem 2.1.11 ([13]). *Let C be a nonempty subset of a normed space X. Then C is closed if and only if the situation $x_n \in C$ and $x_n \to x$ imply that $x \in C$*

Definition 2.1.12 ([13]). Let X be a normed space and a real number $r > 0$. The *closed ball* of X is the set $\{x \in X : \|x\| \leq r\}$ and is denoted by $B_r(0)$. The *unit sphere* of X is the set $\{x \in X : \|x\| = 1\}$ and is denoted by S_X.

Definition 2.1.13 ([13]). A nonempty subset C of a real linear space is *convex* if for each pair of its points, the line segment joining them is contained in C. That is,

$$\{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\} \subset C, \quad \forall x, y \in C.$$

Definition 2.1.14 ([13]). Let X and Y be real linear spaces. A mapping $T : X \to Y$ is called a *linear operator* if

$$T(\alpha x + y) = \alpha T(x) + T(y), \quad \forall x, y \in X \text{ and } \alpha \in \mathbb{R}.$$

Definition 2.1.15 ([13]). Let X be a real linear space. A *linear functional* f on X is a linear operator from X into $\mathbb{R}$.

Proposition 2.1.16 ([17]). Let X be a normed space. Then $\big|\|x\| - \|y\|\big| \le \|x - y\|$ whenever $x, y \in X$. Thus, the function $x \mapsto \|x\|$ is continuous from X into $\mathbb{R}$.

Definition 2.1.17 ([13]). Let X be a normed space. A linear functional $f : X \to \mathbb{R}$ is said to be *bounded* if there exists $M > 0$ such that

$$|f(x)| \le M\|x\|, \quad \forall x \in X.$$

Definition 2.1.18 ([13]). Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} |f(x)|$$

which is called *dual space* of X and denoted by X^*. The *second dual space* of X is the dual space $(X^*)^*$ of X^* and is denoted by X^{**}.

Definition 2.1.19 ([13]). Let X be a normed space. For each $x \in X$ corresponds to a unique bounded linear functional $g_x \in X^{**}$ given by $g_x(f) = f(x)$ ($f \in X^*$ variable). A mapping $C : X \to X^{**}$ defined by $x \mapsto g_x$, C is called the *canonical mapping.*

Definition 2.1.20 ([13]). A normed space X is said to be *reflexive* if the canonical mapping $C : X \to X^{**}$ is *surjective.*

Definition 2.1.21 ([13]). A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there is an $x \in X$ such that for every $f \in X^*$,

$$\lim_{n \to \infty} f(x_n) = f(x).$$

This is written $x_n \rightharpoonup x$ or $x_n \xrightarrow{w} x$. The element x is called the *weak limit of* $\{x_n\}$, and we say that $\{x_n\}$ *converges weakly to* x.

Theorem 2.1.22 ([13, Theorem 4.8-4]). *Let X be a Banach space. If $\{x_n\}$ is a strongly convergent sequence, then it is weakly convergent but the converse is not true.*

Theorem 2.1.23 ([17, Theorem 1.10.7]). *A normed space is reflexive if and only if each of its bounded sequences has a weakly convergent subsequence.*

2.2 Inner product and Hilbert spaces

Definition 2.2.1 ([29]). A real linear space X is called an *inner product space* if there is a real valued function $\langle \cdot, \cdot \rangle$ defined on $X \times X$ with the following properties for each $x, y \in X$ and each scalar α :

(IP1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(IP2) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry),

(IP3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(IP4) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

$\langle x, y \rangle$ is called the *inner product* of x and y.

Proposition 2.2.2 ([29]). Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then $(X, \| \cdot \|)$ is a normed space with a norm defined by

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{2.1}$$

Proposition 2.2.3 ([28]). Let X be an inner product space. If x and y are any two elements in X, then the following statements hold:

(i) $\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$;

(ii) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$;

(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;

(iv) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$ for all $\alpha \in [0, 1]$.

Definition 2.2.4. An inner product space is called a *Hilbert space* if it is a Banach space with the norm defined by (2.1).

Theorem 2.2.5 ([13, Theorem 4.6-6]). *A Hilbert space is reflexive.*

Theorem 2.2.6 ([29, Theorem 5.1.10]). *Let $\{x_n\}$ and $\{y_n\}$ are sequences in an inner product space X and $x, y \in X$. If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.*

Theorem 2.2.7 ([29]). *Let C be a nonempty closed convex subset of a Hilbert H and let $x \in H$. Then there exists a unique element $x_0 \in C$ with*

$$\|x - x_0\| = d(x, C),$$

where $d(x, C) = \inf\{\|x - y\| : y \in C\}$.

Definition 2.2.8 ([29, Definition 5.2.1]). Let C be a nonempty closed convex subset of H and $P : H \to C$ be a mapping. Then for each $x \in H$, there exists a unique element $Px \in C$ such that $\|x - Px\| = d(x, C)$. Such a mapping P of H onto C is called the *metric projection* onto C and denoted by P_C.

Theorem 2.2.9 ([29, Theorem 5.2.3]). *Let C be a nonempty closed convex subset of a Hilbert H and let P_C be the metric projection onto C. Then the followings hold:*

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ *for every* $x, y \in H$;

(ii) *if* $x_n \rightharpoonup x_0$ *and* $P_C x_n \to y_0$, *then* $P_C x_0 = y_0$;

(iii) *if* $x \in H$ *and* $z \in C$, *then* $z = P_C x$ *if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

Definition 2.2.10 ([13]). An linear operator A on an inner product space X is called *strongly positive bounded* if there is $\bar{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2$ for all $x \in X$.

Lemma 2.2.11 ([16]). *Let A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Definition 2.2.12. Let C be a nonempty subset an inner product space X and $\bar{\gamma} > 0$. A mapping A of C into X is said to be

(i) *monotone* if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in C;$$

(ii) $\bar{\gamma}$ *-strongly monotone* if

$$\langle x - y, Ax - Ay \rangle \geq \bar{\gamma}\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) $\bar{\gamma}$*-inverse-strongly monotone* if

$$\langle x - y, Ax - Ay \rangle \geq \bar{\gamma}\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

2.3 Some known facts

Definition 2.3.1. Let C be a nonempty subset of a normed space X and $L > 0$. A mapping $T : C \to X$ is said to be

(i) L-Lipschitzian continuous if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C;$$

(ii) L-contraction if it is L-Lipschitzian continuous with $L < 1$.

Lemma 2.3.2 ([22]). *Let C be a nonempty closed convex subset of a Hilbert space H, T be a quasi-nonexpansive mapping of C into H and $\omega \in (0, 1)$. Then the mapping*

$$T_\omega := (1 - \omega)I + \omega T$$

is strongly quasi-nonexpansive and $F(T) = F(T_w)$.

Definition 2.3.3. Let C be a nonempty subset of a normed space X. A mapping $T : C \to X$ is said to be *demi-closed* at a point $p \in C$ if whenever $\{x_n\}$ is a sequence in C which converges weakly to a point $x \in C$ and $\{Tx_n\}$ converges strongly to p, it follows that $Tx = p$.

Theorem 2.3.4 ([10]). *Let C be a nonempty closed convex subset of a Hilbert space H and T be a nonexpansive mapping of C into H. Then $I - T$ is demiclosed at zero.*

Definition 2.3.5. Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a sequence of mappings of C into H. The set of *common fixed point* of $\{T_n\}$ is denote by $F(\{T_n\})$, that is, $F(\{T_n\}) = \bigcap_{n=1}^{\infty} F(T_n)$.

Definition 2.3.6. Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a sequence of mappings of C into H such that $F(\{T_n\}) \neq \varnothing$. A sequence $\{T_n\}$ is said to be

(i) strongly nonexpansive ([7]) if each T_n is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (T_n x_n - T_n y_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\lim_{n\to\infty} \left(\|x_n - y_n\| - \|T_n x_n - T_n y_n\|\right) = 0;$$

(ii) strongly quasi-nonexpansive ([22]) if each T_n is quasi-nonexpansive and

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$$

whenever $\{x_n\}$ is a bounded sequence in C such that

$$\lim_{n\to\infty} \|x_n - z\| - \|T_n x_n - z\| = 0,$$

for some $z \in F(\{T_n\})$.

Remark 2.3.7. It follows directly from the definition above that if T_n is strongly nonexpansive with a nonempty fixed point set, then it is strongly quasi-nonexpansive.

Definition 2.3.8. Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a sequence of mappings of C into H such that $F(\{T_n\}) \neq \varnothing$. A sequence $\{T_n\}$ is said to satisfy

(i) the *AKTT-condition*([3]) if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in B\} < \infty;$$

(ii) the *NST-condition* ([19]) if for each bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$$

implies $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$;

(iii) the *R-condition* ([1]) if for each bounded subset B of C,

$$\lim_{n\to\infty} \sup_{x\in B}\{\|T_{n+1}x - T_n x\| : x \in B\} = 0.$$

Lemma 2.3.9 ([4, Lemma 3.2]). *Let C be a nonempty closed convex subset of H, $\{T_n\}$ be a family of mappings of C into itself which satisfies the $AKTT$-condition, then the mapping $T : C \to H$ defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in C, \tag{2.2}$$

satisfies

$$\lim_{n\to\infty} \sup_{x\in B}\{\|Tx - T_n x\| : x \in B\} = 0.$$

for each bounded subset B of C.

From now on, we will write $(\{T_n, T\})$ satisfies AKTT-condition and T is defined by (2.2).

Lemma 2.3.10 ([34, Lemma 2.5]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into H such that $F(\{T_n\}) \neq \varnothing$. Suppose that $(\{T_n, T\})$ satisfies $AKTT$-condition, $F(T) = F(\{T_n\})$ and $I - T$ is demi-closed at 0. Then $\{T_n\}$ satisfies the NST-condition and R-condition.*

Lemma 2.3.11 ([4, Corollary 3.13]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of C into H satisfying NST-condition and $F(\{T_n\}) \neq \varnothing$. Let $\{S_n\}$ be a sequence of mappings of C itself defined by*

$$S_n = \beta_n I + (1 - \beta_n)T_n$$

for all $n \in \mathbb{N}$, where $\{\beta_n\}$ is a sequence in $[a, b] \subset (0, 1)$. Then $\{S_n\}$ is a strongly quasi-nonexpansive sequence and $F(\{T_n\}) = F(\{S_n\})$.

Lemma 2.3.12 ([4, Theorem 3.4]). *Let C and K be a nonempty closed convex subset of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of C into K and $\{R_n\}$ be a strongly nonexpansive sequence of K into H such that $F(\{S_n\}) \cap F(\{R_n\}) \neq \varnothing$ Let T_n be a mappings of K into H defined by*

$$T_n = S_n R_n, \quad \forall n \in \mathbb{N}.$$

Then $\{T_n\}$ is a strongly nonexpansive sequence.

Lemma 2.3.13 ([35, Lemma 2.10]). *Let C and K be a nonempty closed convex subset of a Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into K and $\{R_n\}$ be a sequence of nonexpansive mappings of K into H such that $F(\{S_n\}) \cap F(\{R_n\}) \neq \varnothing$ and*

$$\|R_n x - u\|^2 \leq \|x - u\|^2 - a_n \|R_n x - x\|^2$$

for all $x \in K$, $u \in F(\{R_n\})$ and $n \in \mathbb{N}$, where $\{a_n\}$ is a sequence in $[a, \infty) \subset (0, \infty)$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of K into H defined by

$$T_n = S_n R_n, \quad \forall n \in \mathbb{N}.$$

If $\{S_n\}$ and $\{R_n\}$ satisfying NST-condition, then $\{T_n\}$ satisfying NST-condition and $F(\{T_n\}) = F(\{S_n\}) \cap F(\{R_n\})$.

By Lemma 2.3.11 and Lemma 5.6 [35], we have the following.

Lemma 2.3.14. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that $F(\{T_n\}) \neq \varnothing$ and $\{\mu_n\}$ be a sequence in $[0, 1]$. For each $n \in \mathbb{N}$, a W-mapping ([30]) T_n generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$ is defined as follows:*

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \mu_n T_n U_{n,n+1} + (1 - \mu_n) I, \\
&U_{n,n-1} = \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1}) I, \\
&\vdots \\
&U_{n,k} = \mu_k T_k U_{n,k+1} + (1 - \mu_k) I, \\
&U_{n,k-1} = \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1}) I, \\
&\vdots \\
&U_{n,2} = \mu_2 T_2 U_{n,3} + (1 - \mu_2) I, \\
&W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1 - \mu_1) I.
\end{aligned}$$

If $\{\mu_n\} \subset (0, 1)$, then $\{W_n\}$ is a strongly nonexpansive sequence satisfying NST-condition and $F(\{W_n\}) = F(\{T_n\})$.

Definition 2.3.15 ([28]). Let B be a mapping of H into 2^H, where 2^H denotes the set of all subsets of H. A mapping B is said to be *multi-valued mapping* on H.

(i) The *effective domain* of B is the set

$$\{x \in H : Bx \neq \varnothing\}$$

and is denoted by $\mathrm{dom}(B)$.

(ii) The *set of zero points* of B is the set

$$\{x \in \mathrm{dom}(B) : 0 \in Bx\}$$

and is denoted by $B^{-1}(0)$.

Definition 2.3.16. A multi-valued mapping B on H is said to be a *monotone operator* ([5]) on H if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall x, y \in \mathrm{dom}(B), u \in Bx \quad \text{and} \quad v \in By.$$

A monotone operator B on H is said to be *maximal* if its graph is not property contained in the graph of any other monotone operator B' on H.

Lemma 2.3.17 ([2, Example 4.2]). *Let B be a maximal monotone operator on H with a zero point, $\{\rho_n\}$ a sequence of positive real number, and $\{T_n\}$ be a sequence defined by*

$$T_n = (I + \rho_n B)^{-1}, \quad \forall n \in \mathbb{N}.$$

Then $\{T_n\}$ is a strongly nonexpansive sequence. Moreover, if $\liminf_{n\to\infty} \rho_n > 0$, then $\{T_n\}$ satisfies NST-condition and $F(\{T_n\}) = B^{-1}(0)$.

Lemma 2.3.18 ([2, Example 4.3]). *Let $A : C \to H$ be an α- inverse-strongly monotone mapping such that $VI(C, A)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers such that*

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha \tag{2.3}$$

and $\{T_n\}$ a sequence of mappings defined by $T_n = P_C(I - \lambda_n A)$ for $n \in \mathbb{N}$. Then $\{T_n\}$ is a strongly nonexpansive sequence that satisfies NST-condition and $F(\{T_n\}) = VI(C, A)$.

Lemma 2.3.19 ([2, Example 4.4]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $B : C \to H$ be an $\alpha-$inverse-strongly monotone mapping and E be a maximal monotone operator on H. Suppose that $(B+E)^{-1}(0)$ is nonempty and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $0 < \inf_n \lambda_n \leq \sup_n \lambda_n < 2\alpha$. Let $\{T_n\}$ be a sequence of mappings defined by*

$$T_n = (I + \lambda_n E)^{-1}(I - \lambda_n B), \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Then $\{T_n\}$ is a strongly nonexpansive sequence satisfying NST-condition and $F(\{T_n\}) = (B+E)^{-1}(0)$.

Lemma 2.3.20 ([25, Lemma 2.6]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\{\delta_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n \delta_n, \text{ for all } n \in \mathbb{N}.$$

If $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying

$$\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} s_n = 0$.

CHAPTER III

MAIN RESULTS

In this chapter, we introduce a new general iterative algorithm for two quasi-nonexpansive sequences satisfying certain conditions and present strong convergence theorems in which the iteration converges to a common fixed point of these mappings in a Hilbert space. To this end, the following iterative algorithm is introduced:

A sequence $\{x_n\}$ in C defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1-\beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where C is a closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ are two quasi-nonexpansive sequences of C into H such that $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f is a κ - Lipschitzian continuous operator on H with $\kappa > 0$, A is an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$ - contraction for every $0 < \rho < \xi$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\alpha_n < 1/\eta$, and $0 < \gamma < \eta/\kappa$.

3.1 Auxiliary results

In this section, we give lemmas which are needed for proving the our main results.

Lemma 3.1.1. *Let $\{x_n\}$ be a sequence defined by* (3.1). *Then $\{x_n\}$, $\{T_n x_n\}$ and $\{S_n y_n\}$ are bounded.*

Proof. Let $z \in F$. For any $n \in \mathbb{N}$, both S_n and T_n are quasi-nonexpansive, we have

$$\begin{aligned} \|y_n - z\| &= \|P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n\big) - P_C z\| \\ &\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n - z\| \\ &= \|\alpha_n \gamma (f(x_n) - f(z)) + \alpha_n(\gamma f(z) - Az) + (I - \alpha_n A)T_n x_n - (I - \alpha_n A)z\| \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\|f(x_n)-f(z)\|+\alpha_n\|\gamma f(z)-Az\|+\|(I-\alpha_nA)T_nx_n-(I-\alpha_nA)z\| \\
&\leq \alpha_n\gamma\kappa\|x_n-z\|+\alpha_n\|\gamma f(z)-Az\|+(1-\alpha_n\eta)\|T_nx_n-z\| \\
&\leq \alpha_n\gamma\kappa\|x_n-z\|+\alpha_n\|\gamma f(z)-Az\|+(1-\alpha_n\eta)\|x_n-z\| \\
&= (1-\alpha_n(\eta-\gamma\kappa))\|x_n-z\|+\alpha_n\|\gamma f(z)-Az\|. \qquad (3.2)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\|x_{n+1}-z\| \\
&\quad = \|P_C\big(\beta_nx_n+(1-\beta_n)S_ny_n\big)-P_Cz\| \\
&\quad \leq \beta_n\|x_n-z\|+(1-\beta_n)\|S_ny_n-z\| \\
&\quad \leq \beta_n\|x_n-z\|+(1-\beta_n)\|y_n-z\| \\
&\quad \leq \beta_n\|x_n-z\|+(1-\beta_n)\big((1-\alpha_n(\eta-\gamma\kappa))\|x_n-z\|+\alpha_n\|\gamma f(z)-Az\|\big) \\
&\quad = (1-\alpha_n(1-\beta_n)(\eta-\gamma\kappa))\|x_n-z\|+\alpha_n(1-\beta_n)(\eta-\gamma\kappa)\frac{\|\gamma f(z)-Az\|}{\eta-\gamma\kappa}.
\end{aligned}
$$

Let

$$
M=\max\left\{\|x_1-z\|,\frac{\|\gamma f(z)-Az\|}{\eta-\gamma\kappa}\right\}.
$$

Clearly, $\|x_1-z\|\leq M$. Assume that $\|x_k-z\|\leq M$ for some $k\geq 1$. Since $\alpha_n<1/\eta$ and $0<\gamma<\eta/\kappa$, we get $0\leq\alpha_n(1-\beta_n)(\eta-\gamma\kappa)<1$ and so

$$
\|x_{k+1}-z\|\leq(1-\alpha_k(1-\beta_k)(\eta-\gamma\kappa))M+\alpha_k(1-\beta_k)(\eta-\gamma\kappa)M=M.
$$

By induction, we have $\|x_n-z\|\leq M$ for all $n\in\mathbb{N}$ and hence $\{x_n-z\}$ is bounded. It follows that $\{x_n\}$, $\{T_nx_n\}$ and $\{S_ny_n\}$ are also bounded. □

Lemma 3.1.2. *Let $\{x_n\}$ be a sequence defined by (3.1), $z\in F$ and $M=\sup\{\|x_n-z\|:n\in\mathbb{N}\}$. Then*

$$
\|x_{n+1}-z\|^2\leq(1-\gamma_n)\|x_n-z\|^2+\gamma_n\delta_n,\quad\forall n\in\mathbb{N},
$$

where $\gamma_n=\alpha_n(1-\beta_n)(\eta-\gamma\kappa)$ and

$$
\delta_n \;=\; \alpha_n\left(\frac{\eta^2}{(\eta-\gamma\kappa)}-2\kappa\gamma\right)M^2+\frac{2\alpha_n\kappa\gamma M\|\gamma f(z)-Az\|}{\eta-\gamma\kappa}+\frac{2\langle(\gamma f-A)z,y_n-z\rangle}{\eta-\gamma\kappa}.
$$

Proof. From Proposition 2.2.3 (iii) and (3.2), we get

$$
\begin{aligned}
&\|y_n - z\|^2 \\
&\quad = \|P_C(\alpha_n\gamma f(x_n) + (I-\alpha_n A)T_n x_n) - P_C z\|^2 \\
&\quad \le \|\alpha_n(\gamma f(x_n) - Az) + (I-\alpha_n A)T_n x_n - (I-\alpha_n A)z\|^2 \\
&\quad \le \|(I-\alpha_n A)T_n x_n - (I-\alpha_n A)z\|^2 + 2\alpha_n\langle \gamma f(x_n) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|T_n x_n - z\|^2 + 2\alpha_n\langle \gamma f(x_n) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|x_n - z\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(z), y_n - z\rangle \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|x_n - z\|^2 + 2\alpha_n\gamma\|f(x_n) - f(z)\|\|y_n - z\| \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|x_n - z\|^2 + 2\alpha_n\gamma\kappa\|x_n - z\|\|y_n - z\| \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|x_n - z\|^2 + 2\alpha_n\gamma\kappa M\|y_n - z\| + 2\alpha_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\quad \le (1-\alpha_n\eta)^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\gamma\kappa M\Big((1-\alpha_n(\eta-\gamma\kappa))\|x_n - z\| + \alpha_n\|\gamma f(z) - Az\|\Big) \\
&\quad + 2\alpha_n\langle(\gamma f - A)z, y_n - z\rangle \\
&\quad \le \big(1-2\alpha_n(\eta-\gamma\kappa)\big)\|x_n - z\|^2 \\
&\quad + 2\alpha_n(\eta-\gamma\kappa) \\
&\quad \left(\frac{\alpha_n\eta^2M^2}{2(\eta-\gamma\kappa)} - \alpha_n\kappa\gamma M^2 + \frac{\alpha_n\kappa\gamma M\|\gamma f(z) - Az\|}{\eta-\gamma\kappa} + \frac{\langle(\gamma f - A)z, y_n - z\rangle}{\eta-\gamma\kappa}\right) \\
&\quad \le \big(1-\alpha_n(\eta-\gamma\kappa)\big)\|x_n - z\|^2 \\
&\quad + \alpha_n(\eta-\gamma\kappa) \\
&\quad \left(\frac{\alpha_n\eta^2M^2}{(\eta-\gamma\kappa)} - 2\alpha_n\kappa\gamma M^2 + \frac{2\alpha_n\kappa\gamma M\|\gamma f(z) - Az\|}{\eta-\gamma\kappa} + \frac{2\langle(\gamma f - A)z, y_n - z\rangle}{\eta-\gamma\kappa}\right) \\
&\quad = \big(1-\alpha_n(\eta-\gamma\kappa)\big)\|x_n - z\|^2 + \alpha_n(\eta-\gamma\kappa)\delta_n. \qquad (3.3)
\end{aligned}
$$

It follows from Proposition 2.2.3 (iv) that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \|\beta_n(x_n - z) + (1-\beta_n)(S_n y_n - z)\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|S_n y_n - z\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 \\
&\quad +(1-\beta_n)\Big(\big(1-\alpha_n(\eta-\gamma\kappa)\big)\|x_n - z\|^2 + \alpha_n(\eta-\gamma\kappa)\delta_n\Big) \\
&= (1-\alpha_n(1-\beta_n)(\eta-\gamma\kappa))\|x_n - z\|^2 + \alpha_n(1-\beta_n)(\eta-\gamma\kappa)\delta_n \\
&= (1-\gamma_n)\|x_n - z\|^2 + \gamma_n\delta_n.
\end{aligned}
$$

This completes the proof. □

Lemma 3.1.3. *Let $\{x_n\}$ be a sequence defined by (3.1). Suppose, in addition, that $\lim_{n\to\infty}\alpha_n = 0, \limsup_{n\to\infty}\beta_n < 1$ and there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in F$ such that*

$$\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0.$$

Then the following statements hold.

(i) *If $\{T_n\}$ is a strongly quasi-nonexpansive sequence, then*

$$\lim_{k\to\infty}\|T_{n_k}x_{n_k} - x_{n_k}\| = 0.$$

(ii) *If $\{S_n\}$ is a strongly quasi-nonexpansive sequence, then*

$$\lim_{k\to\infty}\|S_{n_k}y_{n_k} - y_{n_k}\| = 0.$$

Proof. Since $\lim_{n\to\infty}\alpha_n = 0$, we have

$$
\begin{aligned}
0 &\leq \liminf_{k\to\infty}\big(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|\big) \\
&= \liminf_{k\to\infty}\big((1-\beta_{n_k})(\|S_{n_k}y_{n_k} - z\| - \|x_{n_k} - z\|)\big) && (3.4) \\
&\leq \liminf_{k\to\infty}(1-\beta_{n_k})\big(\|y_{n_k} - z\| - \|x_{n_k} - z\|\big) && (3.5) \\
&\leq \liminf_{k\to\infty}(1-\beta_{n_k})\big(\alpha_{n_k}\|\gamma f(x_{n_k}) - Az\| + (1-\alpha_{n_k}\eta)\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|\big) \\
&= \lim_{k\to\infty}\alpha_{n_k}(1-\beta_{n_k})\big(\|\gamma f(x_{n_k}) - Az\| - \eta\|T_{n_k}x_{n_k} - z\|\big) \\
&\quad + \liminf_{k\to\infty}(1-\beta_{n_k})\big(\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|\big) \\
&= \liminf_{k\to\infty}(1-\beta_{n_k})\big(\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|\big) \\
&\leq \limsup_{k\to\infty}(1-\beta_{n_k})\big(\|T_{n_k}x_{n_k} - z\| - \|x_{n_k} - z\|\big) \\
&\leq 0. && (3.6)
\end{aligned}
$$

This implies that

$$\lim_{k\to\infty}(1-\beta_{n_k})(\|x_{n_k}-z\|-\|T_{n_k}x_{n_k}-z\|)=0.$$

Since $\limsup_{n\to\infty}\beta_n<1$, we get

$$\lim_{k\to\infty}(\|x_{n_k}-z\|-\|T_{n_k}x_{n_k}-z\|)=0. \tag{3.7}$$

From (3.4), (3.5) and (3.6), we obtain,

$$\lim_{k\to\infty}\big(\|y_{n_k}-z\|-\|x_{n_k}-z\|\big)=0=\lim_{k\to\infty}\big(\|S_{n_k}y_{n_k}-z\|-\|x_{n_k}-z\|\big). \tag{3.8}$$

It follows that

$$\lim_{k\to\infty}\big(\|S_{n_k}y_{n_k}-z\|-\|y_{n_k}-z\|\big)=0. \tag{3.9}$$

(i) Suppose that $\{T_n\}$ is a strongly quasi-nonexpansive sequence. Setting

$$\tilde{x}_n=\begin{cases} x_{n_k} & \text{if } n=n_k \text{ for some } k,\\ z & \text{otherwise.}\end{cases}$$

Since $\{x_{n_k}\}$ is a bounded sequence and (3.7), $\{\tilde{x}_n\}$ is bounded and

$$\lim_{n\to\infty}(\|\tilde{x}_n-z\|-\|T_n\tilde{x}_n-z\|)=0.$$

Since $\{T_n\}$ is a strongly quasi-nonexpansive sequence,

$$\lim_{n\to\infty}\|\tilde{x}_n-T_n\tilde{x}_n\|=0.$$

In particular,

$$\lim_{k\to\infty}\|x_{n_k}-T_{n_k}x_{n_k}\|=0.$$

By using a similar method and (3.9), one can shown that (ii) is satisfied. □

Lemma 3.1.4. *Let $\{x_n\}$ be a sequence defined by (3.1). Suppose, in addition, that $\lim_{n\to\infty}\alpha_n=0, 0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$ and there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z\in F$ such that*

$$\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\ge 0.$$

If $\{T_n\}$ or $\{S_n\}$ is a strongly quasi-nonexpansive sequence, then

$$\lim_{k\to\infty}\|T_{n_k}x_{n_k}-x_{n_k}\|=0 \quad \textit{and} \quad \lim_{k\to\infty}\|S_{n_k}y_{n_k}-y_{n_k}\|=0.$$

Proof. We shall show that $\lim_{k\to\infty}\|S_{n_k}y_{n_k}-x_{n_k}\|=0$. From (3.6), we obtain

$$\lim_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)=0.$$

Since

$$\begin{aligned}\|x_{n_k+1}-z\|^2 &= \beta_{n_k}\|x_{n_k}-z\|^2+(1-\beta_{n_k})\|S_{n_k}y_{n_k}-z\|^2\\ &\quad -\beta_{n_k}(1-\beta_{n_k})\|S_{n_k}y_{n_k}-x_{n_k}\|^2,\end{aligned}$$

we have

$$\begin{aligned}\beta_{n_k}(1-\beta_{n_k})\|S_{n_k}y_{n_k}-x_{n_k}\|^2 &= \beta_{n_k}\big(\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2\big)\\ &\quad +(1-\beta_{n_k})\big(\|S_{n_k}y_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2\big).\end{aligned}$$

Then, by (3.9),

$$\lim_{k\to\infty}\beta_{n_k}(1-\beta_{n_k})\|S_{n_k}y_{n_k}-x_{n_k}\|=0.$$

Since $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, we get,

$$\lim_{k\to\infty}\|S_{n_k}y_{n_k}-x_{n_k}\|=0. \tag{3.10}$$

Case 1: Let $\{T_n\}$ be a strongly quasi-nonexpansive sequence. From Lemma 3.1.3 (i), we have

$$\lim_{k\to\infty}\|T_{n_k}x_{n_k}-x_{n_k}\|=0.$$

It follows from $\alpha_n\to 0$ that

$$\|y_{n_k}-x_{n_k}\| \le \alpha_{n_k}\|\gamma f(x_{n_k})-Ax_{n_k}\|+(1-\alpha_{n_k}\eta)\|T_{n_k}x_{n_k}-x_{n_k}\|\to 0. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\|S_{n_k}y_{n_k}-y_{n_k}\|\le\|S_{n_k}y_{n_k}-x_{n_k}\|+\|x_{n_k}-y_{n_k}\|\to 0. \tag{3.12}$$

Case 2: Let $\{S_n\}$ be a strongly quasi-nonexpansive sequence. Form Lemma 3.1.3 (ii), we have

$$\lim_{k\to\infty}\|S_{n_k}y_{n_k}-y_{n_k}\|=0.$$

It follows from $\alpha_n\to 0$ that

$$\begin{aligned}\|T_{n_k}x_{n_k}-S_{n_k}y_{n_k}\| &\le \|T_{n_k}x_{n_k}-y_{n_k}\|+\|y_{n_k}-S_{n_k}y_{n_k}\|\\ &\le \alpha_{n_k}\|AT_{n_k}x_{n_k}-\gamma f(x_{n_k})\|+\|y_{n_k}-S_{n_k}y_{n_k}\|\to 0. \quad (3.13)\end{aligned}$$

From (3.10) and (3.13), we obtain

$$\|T_{n_k}x_{n_k} - x_{n_k}\| \leq \|T_{n_k}x_{n_k} - S_{n_k}y_{n_k}\| + \|S_{n_k}y_{n_k} - x_{n_k}\| \to 0. \tag{3.14}$$

This completes the proof. $\square$

The following lemmas are extracted from Lemmas 5 and 6[14]. The proof is given here for sake of completeness.

Lemma 3.1.5. *Let C be a nonempty closed convex subset of H. Let A be an operator on H with $\rho > 0$ such that $I - \rho A$ is a contraction and let $w \in C$. Then the followings are equivalent:*

(i) $w = P_C(I - \rho A)w$;

(ii) $\langle Aw, y - w\rangle \geq 0$ *for all* $y \in C$;

(iii) $w = P_C(I - A)w$.

Such $w \in C$ always exists and is unique.

Proof. We have that for $w \in C$,

$$\begin{aligned} w = P_C(I - \rho A)w &\iff \langle w - \rho Aw - w, w - y\rangle \geq 0, \quad \forall y \in C \\ &\iff \langle -\rho Aw, w - y\rangle \geq 0, \quad \forall y \in C \\ &\iff \langle Aw, y - w\rangle \geq 0, \quad \forall y \in C \\ &\iff \langle w - Aw - w, w - y\rangle \geq 0, \quad \forall y \in C \\ &\iff w = P_C(I - A)w. \end{aligned}$$

Then (i), (ii), and (iii) are equivalent. Since $I - \rho A$ is a contraction, we see that $P_C(I - \rho A)$ is a contraction. Therefore such $w \in C$ exists always and is unique. $\square$

Lemma 3.1.6. *Let f be a κ - Lipschitzian continuous operator on H, A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$-contraction and $0 < \gamma < \eta/\kappa$. Then there exist $\bar{\eta} > 0$ and $0 < \bar{\xi} \leq 1/\bar{\eta}$ such that $I - \rho(A - \gamma f)$ is an $(1 - \rho\bar{\eta})$-contraction for every $0 < \rho < \bar{\xi}$. Furthermore, let C be a nonempty closed convex subset of H. Then $P_C(I - A + \gamma f)$ has a unique fixed point $w \in C$. This point $w \in C$ is also a unique solution of the variational inequality*

$$\langle(\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in C.$$

Proof. Let $\bar{\eta} = \eta - \gamma\kappa$ and $\bar{\xi} = \xi$. Then $0 < \bar{\xi} \leq 1/\bar{\eta}$. For $0 < \rho < \bar{\xi}$, we get

$$\begin{aligned}
\|(I - \rho(A - \gamma f))x - (I-\rho(A - \gamma f))y\| \\
&\leq \|(I - \rho A)x - (I - \rho A)y + \rho\gamma(f(x) - f(y))\| \\
&\leq \|(I - \rho A)x - (I - \rho A)y\| + \|\rho\gamma(f(x) - f(y))\| \\
&\leq (I - \rho\eta)\|x - y\| + \rho\gamma\kappa\|x - y\| \\
&= (1 - \rho(\eta - \gamma\kappa))\|x - y\| \\
&= (1 - \rho\bar{\eta})\|x - y\|
\end{aligned}$$

for all $x, y \in H$. Therefore, $I - \rho(A - \gamma f)$ is an $(1 - \rho\bar{\eta})$ - contraction. Using Lemma 3.1.5, we have $P_C(I - A + \gamma f)$ has a unique fixed point $w \in C$. This point $w \in C$ is also a unique solution of the variational inequality

$$\langle(\gamma f - A)w, q - w\rangle \leq 0,$$

for all $q \in C$. $\square$

3.2 Strong convergence theorems

Theorem 3.2.1. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ be two quasi-nonexpansive sequences of C into H satisfying the NST-condition and $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$- contraction for every $0 < \rho < \xi$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \eta/\kappa$. Assume one of the following conditions hold:*

(i) *$\{T_n\}$ or $\{S_n\}$ is a strongly quasi-nonexpansive sequence and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*

(ii) *$\{T_n\}$ and $\{S_n\}$ are strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.*

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to an element $w \in F$ and the following inequality holds,

$$\langle(\gamma f - A)w, p - w\rangle \leq 0, \quad \forall p \in F.$$

Proof. Notice that F is a nonempty closed convex subset of H. By Lemma 3.1.6, there exists a unique element $w \in F$ such that $w = P_F(I - A + \gamma f)(w)$, i.e.,

$$\langle (\gamma f - A)w, p - w \rangle \leq 0, \quad \forall p \in F. \tag{3.15}$$

By Lemma 3.1.2, we have

$$\|x_{n+1} - w\|^2 \leq (1 - \gamma_n)\|x_n - w\|^2 + \gamma_n \delta_n, \tag{3.16}$$

for all $n \in \mathbb{N}$, where $\gamma_n = \alpha_n(1 - \beta_n)(\eta - \gamma\kappa)$, $M = \sup\{\|x_n - w\| : n \in \mathbb{N}\}$ and

$$\delta_n = \alpha_n\left(\frac{\eta^2}{(\eta - \gamma\kappa)} - 2\kappa\gamma\right)M^2 + \frac{2\alpha_n\kappa\gamma M\|\gamma f(w) - Aw\|}{\eta - \gamma\kappa} + \frac{2\langle(\gamma f - A)w, y_n - w\rangle}{\eta - \gamma\kappa}.$$

We show that $x_n \to w$ by using Lemma 2.3.20. Since $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we get $\sum_{n=0}^{\infty} \gamma_n = \infty$ and then we only show that

$$\limsup_{k\to\infty} \delta_{n_k} = \frac{2}{\eta - \gamma\kappa} \limsup_{k\to\infty} \langle (\gamma f - A)w, y_{n_k} - w\rangle \leq 0,$$

for every subsequence $\{n_k\}$ of $\{n\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - w\| - \|x_{n_k} - w\|) \geq 0$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - w\| - \|x_{n_k} - w\|) \geq 0.$$

Assume (i) or (ii). By Lemma 3.1.3 or Lemma 3.1.4, respectively, we have

$$\lim_{k\to\infty} \|T_{n_k}x_{n_k} - x_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|S_{n_k}y_{n_k} - y_{n_k}\| = 0.$$

Because $\{y_{n_k}\}$ is a bounded sequence in H, there exists a subsequence $\{y_{n_{k_\ell}}\}$ of $\{y_{n_k}\}$ such that $y_{n_{k_\ell}} \rightharpoonup q$ and

$$\limsup_{k\to\infty} \langle (\gamma f - A)w, y_{n_k} - w\rangle = \lim_{\ell\to\infty} \langle (\gamma f - A)w, y_{n_{k_\ell}} - w\rangle. \tag{3.17}$$

Since $\{S_n\}$ satisfies the NST- condition, we have $q \in F(\{S_n\})$. Since $\alpha_n \to 0$,

$$\|y_{n_k} - x_{n_k}\| \leq \alpha_{n_k}\|\gamma f(x_{n_k}) - Ax_{n_k}\| + (1 - \alpha_{n_k}\eta)\|T_{n_k}x_{n_k} - x_{n_k}\| \to 0$$

and hence $x_{n_{k_\ell}} \rightharpoonup q$. Since $\{T_n\}$ satisfies NST- condition, we have $q \in F(\{T_n\})$. Then

$$q \in F(\{S_n\}) \cap F(\{T_n\}).$$

It follows from (3.15) and (3.17) that

$$\limsup_{k\to\infty} \langle (\gamma f - A)w, y_{n_k} - w\rangle = \lim_{\ell\to\infty} \langle (\gamma f - A)w, y_{n_{k_\ell}} - w\rangle = \langle (\gamma f - A)w, q - w\rangle \leq 0.$$

This implies that $\limsup_{k\to\infty} \delta_{n_k} \leq 0$. By Lemma 2.3.20 and (3.16) we have $x_n \to w$ as desired. $\square$

Lemma 3.2.2. *Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Let $\eta \in (0, \bar{\gamma}/2]$ and $\xi = \min\{\frac{1}{\eta}, \frac{2\eta}{L^2}\}$. Then $I - \rho A$ is an $(1 - \rho\eta)$- contraction for every $0 < \rho < \xi$.*

Proof. For any $0 < \rho < \xi$ and $x, y \in H$, we have

$$\begin{aligned}
\|(I - \rho A)x - (I - \rho A)y\|^2 &= \|x - y - \rho(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\rho\langle x - y, Ax - Ay\rangle + \rho^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\rho\bar{\gamma}\|x - y\|^2 + \rho^2 L^2\|x - y\|^2 \\
&= (1 - 2\rho\bar{\gamma} + \rho^2 L^2)\|x - y\|^2 \\
&= (1 - 4\rho\eta + \rho^2 L^2)\|x - y\|^2 \\
&\leq (1 - 2\rho\eta + \rho^2\eta^2 - \rho^2\eta^2 - 2\rho\eta + \rho^2 L^2)\|x - y\|^2 \\
&= \left[(1 - \rho\eta)^2 - \rho^2\eta^2 - \rho L^2(\frac{2\eta}{L^2} - \rho)\right]\|x - y\|^2 \\
&\leq (1 - \rho\eta)^2\|x - y\|^2.
\end{aligned}$$

Since $0 < \rho\eta < 1$, we obtain that

$$\|(I - \rho A)x - (I - \rho A)y\| \leq (1 - \rho\eta)\|x - y\|,$$

for all $x, y \in H$. This completes the proof. □

Using Theorem 3.2.1 and Lemma 3.2.2, we have the following.

Corollary 3.2.3. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ be two quasi-nonexpansive sequences of C into H satisfying the NST-condition and $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \bar{\gamma}/2\kappa$. Assume one of the following conditions hold:*

(i) *$\{T_n\}$ or $\{S_n\}$ is a strongly quasi-nonexpansive sequence and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*

(ii) *$\{T_n\}$ and $\{S_n\}$ are strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.*

Then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to an element* $w \in F$ *and the following inequality holds,*

$$\langle(\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in F.$$

As in Lemma 3.2.2, we can relax an η which is widely whenever A is a strongly positive bounded linear operator on H as follows:

Lemma 3.2.4. *Let* A *be a strongly positive bounded linear operator of* H *with* $\bar{\gamma} > 0$. *Let* $\eta \in (0, \bar{\gamma}]$ *and* $\xi = \min\left\{\frac{1}{\eta}, \frac{1}{\|A\|}\right\}$. *Then* $I - \rho A$ *is an* $(1 - \rho\eta)$ *- contraction for every* $0 < \rho < \xi$.

Proof. Let $0 < \rho < \xi$. By Lemma 2.2.11, we get

$$\begin{aligned} \|(I - \rho A)x - (I - \rho A)y\| &= \|(I - \rho A)(x - y)\| \\ &\leq \|I - \rho A\|\|x - y\| \\ &\leq (1 - \rho\eta)\|x - y\|. \end{aligned}$$

This implies that $I - \rho A$ is an $(1 - \rho\eta)$ - contraction. □

By Theorem 3.2.1 and Lemma 3.2.4, we obtain the following result.

Corollary 3.2.5. *Let* C *be a nonempty closed convex subset of a Hilbert space* H, $\{S_n\}$ *and* $\{T_n\}$ *be two quasi-nonexpansive sequences of* C *into* H *satisfying the NST-condition and* $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f *be a* κ *- Lipschitzian continuous operator on* H *with* $\kappa > 0$. *Let* A *be a strongly positive bounded linear operator on* H *with* $\overline{\gamma} > 0$. *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $0 < \gamma < \overline{\gamma}/\kappa$. *Assume one of the following conditions hold:*

(i) $\{T_n\}$ *or* $\{S_n\}$ *is a strongly quasi-nonexpansive sequence and* $\{\beta_n\}$ *is a sequence in* $(0, 1)$ *satisfying* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(ii) $\{T_n\}$ *and* $\{S_n\}$ *are strongly quasi-nonexpansive sequences and* $\{\beta_n\}$ *is a sequence in* $(0, 1)$ *satisfying* $\limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to an element* $w \in F$ *and the following inequality holds,*

$$\langle(\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in F.$$

When $S_n \equiv I$ and $\beta_n \equiv 0$ in Theorem 3.2.1, we have the following result.

Corollary 3.2.6. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a strongly quasi-nonexpansive sequence of C into H satisfying the NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$-contraction for every $0 < \rho < \xi$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \eta/\kappa$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n\big), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.18}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in F(\{T_n\})$ and the following inequality holds,

$$\langle (\gamma f - A)w, q - w \rangle \leq 0, \quad \forall q \in F(\{T_n\}).$$

By Corollary 3.2.6 and Lemma 3.2.2, we obtain the following result.

Corollary 3.2.7. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a strongly quasi-nonexpansive sequence satisfying NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \bar{\gamma}/2\kappa$. Then the sequence $\{x_n\}$ defined by* (3.18) *converges strongly to $w \in F(\{T_n\})$ and the following inequality holds,*

$$\langle (\gamma f - A)w, q - w \rangle \leq 0, \qquad \forall q \in F(\{T_n\}).$$

By Theorem 3.2.6 and Lemma 3.2.4, we obtain the following result.

Corollary 3.2.8. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a strongly quasi-nonexpansive sequence satisfying NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \overline{\gamma}/\kappa$.*

Then the sequence $\{x_n\}$ *defined by* (3.18) *converges strongly to* $w \in F(\{T_n\})$ *and the following inequality holds,*

$$\langle(\gamma f - A)w, q - w\rangle \leq 0, \qquad \forall q \in F(\{T_n\}).$$

Theorem 3.2.9. *Let* C *be a nonempty closed convex subset of a Hilbert space* H, $\{T_n\}$ *be a sequence of quasi-nonexpansive mappings of* C *into* H *satisfying the NST-condition and* $F(\{T_n\}) \neq \varnothing$, f *be a* κ *- Lipschitzian continuous operator on* H *with* $\kappa > 0$. *Let* A *be an operator on* H *with* $\eta > 0$ *and* $0 < \xi \leq 1/\eta$ *such that* $I - \rho A$ *is an* $(1 - \rho\eta)$*-contraction for every* $0 < \rho < \xi$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are a sequences in* $(0, 1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$, *and* $0 < \gamma < \eta/\kappa$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)T_n x_n, \\ x_{n+1} = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n\big), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.19}$$

Then the sequence $\{x_n\}$ *converges strongly to an element* $w \in F(\{T_n\})$ *and the following inequality holds,*

$$\langle(\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in F(\{T_n\}).$$

Proof. Set $\widehat{S}_n \equiv I$, $\widehat{T}_n \equiv \beta_n I + (1 - \beta_n)T_n$, $\widehat{\alpha}_n \equiv \alpha_n$ and $\widehat{\beta}_n \equiv 0$. Define a sequence $\{\widehat{x}_n\}$ by

$$\begin{cases} \widehat{x}_1 = x_1 \in C, \\ \widehat{y}_n = P_C\big(\widehat{\alpha}_n \gamma f(\widehat{x}_n) + (I - \widehat{\alpha}_n A)\widehat{T}_n \widehat{x}_n\big), \\ \widehat{x}_{n+1} = P_C\big(\widehat{\beta}_n \widehat{x}_n + (1 - \widehat{\beta}_n)\widehat{S}_n \widehat{y}_n\big), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.20}$$

By Lemma 2.3.11, $\{\hat{T_n}\}$ is a strongly quasi-nonexpansive sequence and $F(\{\widehat{T}_n\}) = F(\{T_n\})$. Moreover, the iterative schemes (3.19) and (3.20) are equivalent. Applying Theorem 3.2.1, we conclude that $\{x_n\}$ converges strongly to an element $w \in F(\{T_n\})$ and the following inequality holds,

$$\langle(\gamma f - A)w, q - w\rangle \leq 0,$$

for all $q \in F(\{T_n\})$. □

By Theorem 3.2.9 and Lemma 3.2.2, we obtain the following result.

Corollary 3.2.10. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of C into H satisfying NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are a sequences in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$, and $0 < \gamma < \bar{\gamma}/2\kappa$. Then the sequence $\{x_n\}$ defined by (3.19) converges strongly to $w \in F(\{T_n\})$ and the following inequality holds,*

$$\langle (\gamma f - A)w, q - w \rangle \leq 0, \qquad \forall q \in F(\{T_n\}).$$

Remark 3.2.11. When $T_n \equiv T$ and $\beta_n \equiv \omega \in (0,1)$ in Corollary 3.2.10, then result extends and improves [32, Theorem 3.1].

By Theorem 3.2.9 and Lemma 3.2.4, we obtain the following result.

Corollary 3.2.12. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a strongly quasi-nonexpansive sequence satisfying NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are a sequences in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$, and $0 < \gamma < \overline{\gamma}/\kappa$. Then the sequence $\{x_n\}$ defined by (3.19) converges strongly to $w \in F(\{T_n\})$ and the following inequality holds,*

$$\langle (\gamma f - A)w, q - w \rangle \leq 0, \qquad \forall q \in F(\{T_n\}).$$

Remark 3.2.13. When $T_n \equiv T$ and $\beta_n \equiv \omega \in (0,1)$ in Corollary 3.2.12, then result extends and improves [33, Theorem 3.1].

When $A \equiv I$ and $\gamma \equiv 1$ in Theorem 3.2.1, we have the following result.

Theorem 3.2.14. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ be two quasi-nonexpansive sequences of C into H satisfying the NST-condition and $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f be a κ - contraction continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume one of the following conditions hold:*

(i) $\{T_n\}$ *or* $\{S_n\}$ *is a strongly quasi-nonexpansive sequence and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(ii) $\{T_n\}$ *and* $\{S_n\}$ *are strongly quasi-nonexpansive sequences and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1-\beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.21}$$

Then the sequence $\{x_n\}$ *converges strongly to an element* $w \in F$ *and the following inequality holds,*

$$\langle f(w) - w, q - w\rangle \leq 0, \quad \forall q \in F.$$

Since every strongly nonexpansive sequence with a nonempty common fixed point set is strongly quasi-nonexpansive, we obtain the following result.

Corollary 3.2.15. *Let* C *be a nonempty closed convex subset of a Hilbert space* H, $\{S_n\}$ *and* $\{T_n\}$ *be two nonexpansive sequences of* C *into* H *satisfying the NST-condition and* $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f *be a* κ *- Lipschitzian continuous operator on* H *with* $\kappa > 0$. *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Assume one of the following conditions hold:*

(i) $\{T_n\}$ *or* $\{S_n\}$ *is a strongly nonexpansive sequence and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(ii) $\{T_n\}$ *and* $\{S_n\}$ *are strongly nonexpansive sequences and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by* (3.21) *converges strongly to an element* $w \in F$ *and the following inequality holds,*

$$\langle f(w) - w, q - w\rangle \leq 0, \quad \forall q \in F.$$

Remark 3.2.16. Our Corollary 3.2.15 extends and improves [2, Theorem 3.1] in the following way:

(1) The R-condition is removed.

(2) The restriction condition

$$\lim_{n\to\infty}\beta_n = 0 \quad \text{or} \quad 0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$$

is weakened and replaced by $\limsup_{n\to\infty}\beta_n < 1$ whenever $\{S_n\}$ and $\{T_n\}$ are both strongly nonexpansive sequences.

(3) The class of contraction mappings is wider than the class of constant mappings.

When $S_n \equiv I$ and $\beta_n \equiv 0$ in Theorem 3.2.14, we have the following result.

Theorem 3.2.17. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be strongly quasi-nonexpansive sequence of C into H satisfying the NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - contraction continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n x_n\big), \qquad \forall n \in \mathbb{N}. \end{cases} \tag{3.22}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in F(\{T_n\})$ and the following inequality holds,

$$\langle f(w) - w, q - w\rangle \leq 0, \quad \forall q \in F(\{T_n\}).$$

When $T_n \equiv T$ in Theorem 3.2.17, we have the following result.

Corollary 3.2.18 ([37, Theorem 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H, T be a strongly quasi-nonexpansive mapping of C into H such that $I-T$ is a demiclosed at zero and $F(T) \neq \varnothing$, f be a κ - contraction continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T x_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to $w \in F(T)$ and the following inequality holds,

$$\langle f(w) - w, q - w\rangle \leq 0, \quad \forall q \in F(T).$$

3.3 Applications

Let C be a nonempty closed convex subset of a Hilbert space H. Let $g : C \times C \to \mathbb{R}$ be a bifunction and let $A : C \to H$ be a nonlinear mapping. Then, we consider the following *equilibrium problem*:

$$\text{Find } \; z \in C \; \text{ such that } \; g(z, y) + \langle Az, y - z \rangle \geq 0 \; \text{ for all } \; y \in C. \tag{3.23}$$

The set of such z is denote by $EP(g, A)$, i.e.,

$$EP(g, A) = \{z \in C : g(z, y) + \langle Az, y - z \rangle \geq 0 \; \text{ for all } \; y \in C\}.$$

In the case of $A \equiv 0$, $EP(g, A)$ is denoted by $EP(g)$. In the case of $g \equiv 0$, $EP(g, A)$ is denoted by $VI(C, A)$. The equilibrium problem is very general in the sense that it includes, as special case, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games.

For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x, x) = 0$ for all $x \in C$;

(A2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;

(A3) g is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} g(tz + (1 - t)x, y) \leq g(x, y);$$

(A4) $g(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma was given in Combettes and Hirstoaga [9] and Takahashi, Takahashi and Toyoda [27].

Lemma 3.3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $g : C \times C \to \mathbb{R}$ satisfies conditions $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\right\}$$

for all $x \in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) *$F(T_r) = EP(g)$;*

(4) *$EP(g)$ is closed and convex.*

We call such T_r the resolvent of g for $r > 0$.

Lemma 3.3.2 ([35, Lemma 4.4]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $g : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A4)$ and let $r > 0$. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H with $EP(g, A) \neq \varnothing$, then*

$$\|T_r(I - rA)x - u\|^2 \leq \|x - u\|^2 - \frac{2\alpha - r}{2\alpha}\|T_r(I - rA)x - x\|^2$$

for all $x \in C$ and $u \in F(T_r(I - rA)) = EP(g, A)$. Furthermore, if $0 \leq r \leq 2\alpha$, then $T_r(I - rA)$ is a nonexpansive mapping of C into itself.

Lemma 3.3.3 ([35, Lemma 4.5]). *Let C be a nonempty closed convex subset of a Hilbert space H, let g be a bifunction from $H \times H$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let A be an α-inverse-strongly monotone mapping of C into H with $\alpha > 0$ and $EP(g, A) \neq \varnothing$. Let $\{R_n\}$ be a sequence of mappings of C into itself defined by*

$$R_n = T_{r_n}(I - r_n A),$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \to \infty} r_n > 0$. Then $\{R_n\}$ is a strongly nonexpansive sequence satisfying NST-condition and $F(\{R_n\}) = EP(g, A)$.

Now, we apply Theorem 3.2.6 to the following strong convergence theorem for finding a common solution of a common fixed point problem for a sequence of nonexpansive mappings and of an equilibrium problem for a bifunction in a Hilbert space.

Theorem 3.3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let g be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying conditions $(A1) - (A4)$. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$-contraction for every $0 < \rho < \xi$ and B be an α-inverse-strongly monotone mapping of C into H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that $\Omega = F(\{T_n\}) \cap EP(g, B) \neq \varnothing$ and f be a κ- Lipschitzian continuous operator on H with $\kappa > 0$. Let $\{\mu_n\}$ be a sequence in $(0, 1)$. Let W_n be a W-mapping of C into H generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ is a sequence in $[c, d]$ for some $c, d \in (0, 2\alpha)$, and $0 < \gamma < \eta/\kappa$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in C$ and*

$$\begin{cases} g(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n u_n), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.24}$$

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

Proof. Let $R_n \equiv T_{r_n}(I - r_n B)x_n$ and $\widehat{T}_n \equiv W_n R_n$. By Lemmas 2.3.13, 2.3.14 and 3.3.3, we get $\{\widehat{T}_n\}$ is a strongly nonexpansive sequence satisfying NST-condition and $F(\{\widehat{T}_n\}) = F(\{W_n\}) \cap EP(g, B) = F(\{T_n\}) \cap EP(g, B) \neq \varnothing$. Moreover, the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.24) can be rewrite by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)\hat{T}_n x_n), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.25}$$

Applying Theorem 3.2.6, we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

□

By Theorem 3.3.4 and Lemma 3.2.2, we obtain the following result.

Corollary 3.3.5. *Let C be a closed convex subset of a Hilbert space H. Let g be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H and B be an α-inverse-strongly monotone mapping of C into H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that $\Omega = F(\{T_n\}) \cap EP(g, B) \neq \varnothing$ and f be a κ- Lipschitzian continuous operator on H with $\kappa > 0$. Let $\{\mu_n\}$ be a sequence in $(0,1)$. Let W_n be a W-mapping of C into H generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ is a sequence in $[c,d]$ for some $c, d \in (0, 2\alpha)$, and $0 < \gamma < \bar{\gamma}/2\kappa$. Then both $\{x_n\}$ and $\{u_n\}$ defined by (3.24) converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)w, w - q \rangle \geq 0, \quad \forall q \in \Omega.$$

By Theorem 3.3.4 and Lemma 3.2.4, we obtain the following result.

Corollary 3.3.6. *Let C be a closed convex subset of a Hilbert space H. Let g be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let A be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$ and B be an α-inverse-strongly monotone mapping of C into H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that $\Omega = F(\{T_n\}) \cap EP(g, B) \neq \varnothing$ and f be a κ- Lipschitzian continuous operator on H with $\kappa > 0$. Let $\{\mu_n\}$ be a sequence in $(0,1)$. Let W_n be a W-mapping of C into H generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ is a sequence in $[c,d]$ for some $c, d \in (0, 2\alpha)$, and $0 < \gamma < \overline{\gamma}/\kappa$. Then both $\{x_n\}$ and $\{u_n\}$ defined by (3.24) converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)w, w - q \rangle \geq 0, \quad \forall q \in \Omega.$$

When $B \equiv 0$ in Theorem 3.3.4, we have the following result.

Theorem 3.3.7. *Let C be a closed convex subset of a Hilbert space H. Let g be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let A be operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$- contraction for every $0 < \rho < \xi$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that*

$\Omega = F(\{T_n\}) \cap EP(g) \neq \varnothing$ *and* f *be a* κ*- Lipschitzian continuous operator on* H *with* $\kappa > 0$. *Let* $\{\mu_n\}$ *be a sequence in* $(0,1)$. *Let* W_n *be a* W*-mapping of* C *into* H *generated by* $T_n, T_{n-1}, ..., T_1$ *and* $\mu_n, \mu_{n-1}, ..., \mu_1$. *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ *is a sequence in* $[c,d]$ *for some* $c, d \in (0, 2\alpha)$, *and* $0 < \gamma < \eta/\kappa$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 = x \in C$ *and*

$$\begin{cases} g(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n), & \forall n \in \mathbb{N}. \end{cases} \tag{3.26}$$

Then both $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $w \in \Omega$, *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

By Theorem 3.3.7 and Lemma 3.2.2, we obtain the following result.

Corollary 3.3.8. *Let* C *be a closed convex subset of a Hilbert space* H. *Let* g *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1)-(A4)$. *Let* A *be a* $\bar{\gamma}$*-strongly monotone and* L*-Lipschitzian continuous operator on* H *with* $\bar{\gamma} > 0$ *and* $L > 0$. *Let* $\{T_n\}$ *be sequence of nonexpansive mappings of* C *into* H *such that* $\Omega = F(\{T_n\}) \cap EP(g) \neq \varnothing$ *and* f *be a* κ*- Lipschitzian continuous operator on* H *with* $\kappa > 0$. *Let* $\{\mu_n\}$ *be a sequence in* $(0,1)$. *Let* W_n *be a* W*-mapping of* C *into* H *generated by* $T_n, T_{n-1}, ..., T_1$ *and* $\mu_n, \mu_{n-1}, ..., \mu_1$. *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ *is a sequence in* $[c,d]$ *for some* $c, d \in (0, 2\alpha)$, *and* $0 < \gamma < \bar{\gamma}/2\kappa$. *Then both* $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.26) *converge strongly to* $w \in \Omega$, *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

By Theorem 3.3.7 and Lemma 3.2.4, we obtain the following result.

Corollary 3.3.9 ([21, Theorem 3.1]). *Let* C *be a closed convex subset of a Hilbert space* H. *Let* g *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1)-(A4)$. *Let* A *be a strongly positive bounded linear operator on* H *with* $\overline{\gamma} > 0$. *Let* $\{T_n\}$ *be a sequence of nonexpansive mappings of* C *into* H *such that* $\Omega = F(\{T_n\}) \cap EP(g) \neq \varnothing$ *and* f *be a*

κ- Lipschitzian continuous operator on H with $\kappa > 0$. Let $\{\mu_n\}$ be a sequence in $(0,1)$. Let W_n be a W-mapping of C into H generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ is a sequence in $[c,d]$ for some $c, d \in (0, 2\alpha)$, and $0 < \gamma < \overline{\gamma}/\kappa$. Then both $\{x_n\}$ and $\{u_n\}$ defined by (3.26) converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

By Theorem 3.2.1 and Lemma 2.3.19, we obtain the following result which is a strong convergence theorem for finding a common solution of a common fixed point problem for a sequence of quasi-nonexpansive mappings and of a monotone inclusion problem for the sum of two monotone mappings in a Hilbert space.

Theorem 3.3.10. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ be a quasi-nonexpansive sequence of C into H satisfying the NST-condition. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$-contraction for every $0 < \rho < \xi$ and B be an $\alpha-$inverse-strongly monotone mapping of C into H and E be a maximal monotone operator on H such that $\Omega = F(\{S_n\}) \cap (B + E)^{-1}(0) \neq \varnothing$. Let f be a κ - Lipschitzian continuous operator on H. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\}$ be a sequence of positive real numbers satisfying $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, and $0 < \gamma < \eta/\kappa$. Assume one of the following conditions hold:*

(i) *$\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*

(ii) *$\{S_n\}$ is a strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.*

Let $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)((I + \lambda_n E)^{-1}(I - \lambda_n B))x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1 - \beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.27}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in \Omega$ and the following inequality holds,

$$\langle(\gamma f - A)w, p - w\rangle \leq 0, \quad \forall p \in \Omega.$$

By Theorem 3.3.10 and Lemma 3.2.2, we obtain the following result.

Corollary 3.3.11. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ be a quasi-nonexpansive sequence of C into H satisfying the NST-condition. Let A be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Let B be an $\alpha-$inverse-strongly monotone mapping of C into H and E be a maximal monotone operator on H such that $\Omega = F(\{S_n\}) \cap (B+E)^{-1}(0) \neq \varnothing$. Let f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, and $0 < \gamma < \bar{\gamma}/2\kappa$. Assume one of the following conditions hold:*

(i) *$\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*

(ii) *$\{S_n\}$ is a strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.*

Then the sequence $\{x_n\}$ defined by (3.27) converges strongly to an element $w \in \Omega$ and the following inequality holds,

$$\langle(\gamma f - A)w, p - w\rangle \leq 0, \quad \forall p \in \Omega.$$

By Theorem 3.3.10 and Lemma 3.2.4, we obtain the following result.

Corollary 3.3.12. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ be a quasi-nonexpansive sequence of C into H satisfying the NST-condition. Let A be a strongly positive bounded linear operator on H with $\bar{\gamma} > 0$ and B be an $\alpha-$inverse-strongly monotone mapping of C into H, and E be a maximal monotone operator on H such that $\Omega = F(\{S_n\}) \cap (B+E)^{-1}(0) \neq \varnothing$. Let f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, and $0 < \gamma < \bar{\gamma}/\kappa$. Assume one of the following conditions hold:*

(i) $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(ii) $\{S_n\}$ *is a strongly quasi-nonexpansive sequences and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by* (3.27) *converges strongly to an element* $w \in \Omega$ *and the following inequality holds,*

$$\langle (\gamma f - A)w, p - w \rangle \leq 0, \quad \forall p \in \Omega.$$

CHAPTER IV
CONCLUSIONS

We summarize results as follow:

1. Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ be two quasi-nonexpansive sequences of C into H satisfying the NST-condition and $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$- contraction for every $0 < \rho < \xi$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \gamma < \eta/\kappa$. Assume one of the following conditions hold:

   (i) $\{T_n\}$ or $\{S_n\}$ is a strongly quasi-nonexpansive sequence and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

   (ii) $\{T_n\}$ and $\{S_n\}$ are strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.

   Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1 - \beta_n) S_n y_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

   Then the sequence $\{x_n\}$ converges strongly to an element $w \in F$ and the following inequality holds,

$$\langle (\gamma f - A)w, p - w \rangle \leq 0, \quad \forall p \in F.$$

2. Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a strongly quasi-nonexpansive sequence of C into H satisfying the NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an

$(1-\rho\eta)$ - contraction for every $0 < \rho < \xi$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $0 < \gamma < \eta/\kappa$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C\big(\alpha_n\gamma f(x_n) + (I - \alpha_n A)T_n x_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in F(\{T_n\})$ and the following inequality holds,

$$\langle (\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in F(\{T_n\}).$$

3. Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of C into H satisfying the NST-condition and $F(\{T_n\}) \neq \varnothing$, f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1-\rho\eta)$ - contraction for every $0 < \rho < \xi$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are a sequences in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \liminf_{n\to\infty}\beta_n < 1$, and $0 < \gamma < \eta/\kappa$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)T_n x_n, \\ x_{n+1} = P_C\big(\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in F(\{T_n\})$ and the following inequality holds,

$$\langle (\gamma f - A)w, q - w\rangle \leq 0, \quad \forall q \in F(\{T_n\}).$$

4. Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ and $\{T_n\}$ be two quasi-nonexpansive sequences of C into H satisfying the NST-condition and $F := F(\{S_n\}) \cap F(\{T_n\}) \neq \varnothing$, f be a κ - contraction continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Assume one of the following conditions hold:

(i) $\{T_n\}$ or $\{S_n\}$ is a strongly quasi-nonexpansive sequence and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(ii) $\{T_n\}$ and $\{S_n\}$ are strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1-\beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to an element $w \in F$ and the following inequality holds,

$$\langle f(w) - w, q - w\rangle \leq 0, \quad \forall q \in F.$$

5. Let C be a nonempty closed convex subset of a Hilbert space H. Let g be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying conditions $(A1)-(A4)$. Let A be an operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1-\rho\eta)$- contraction for every $0 < \rho < \xi$ and B be an α-inverse-strongly monotone mapping of C into H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H such that $\Omega = F(\{T_n\}) \cap EP(g,B) \neq \varnothing$ and f be a κ-Lipschitzian continuous operator on H with $\kappa > 0$. Let $\{\mu_n\}$ be a sequence in $(0,1)$. Let W_n be a W-mapping of C into H generated by $T_n, T_{n-1}, ..., T_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{r_n\}$ is a sequence in $[c,d]$ for some $c, d \in (0, 2\alpha)$, and $0 < \gamma < \eta/\kappa$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in C$ and

$$\begin{cases} g(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $w \in \Omega$, which is the unique solution of the variational inequality

$$\langle (A - \gamma f)w, w - q\rangle \geq 0, \quad \forall q \in \Omega.$$

6. Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ be a quasi-nonexpansive sequence of C into H satisfying the NST-condition. Let A be operator on H with $\eta > 0$ and $0 < \xi \leq 1/\eta$ such that $I - \rho A$ is an $(1 - \rho\eta)$ - contraction for every $0 < \rho < \xi$ and B be an $\alpha-$inverse-strongly monotone mapping of C into H, and E be a maximal monotone operator on H such that $\Omega = F(\{S_n\}) \cap (B + E)^{-1}(0) \neq \varnothing$. Let f be a κ - Lipschitzian continuous operator on H with $\kappa > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, and $0 < \gamma < \eta/\kappa$. Assume one of the following conditions hold:

   (i) $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying
   $$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

   (ii) $\{S_n\}$ is a strongly quasi-nonexpansive sequences and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying $\limsup_{n\to\infty} \beta_n < 1$.

   Let $\{x_n\}$ defined by
   $$\begin{cases} x_1 = x \in C, \\ y_n = P_C\big(\alpha_n\gamma f(x_n) + (I - \alpha_n A)((I + \lambda_n E)^{-1}(I - \lambda_n B))x_n\big), \\ x_{n+1} = P_C\big(\beta_n x_n + (1 - \beta_n)S_n y_n\big), \quad \forall n \in \mathbb{N}. \end{cases}$$

   Then the sequence $\{x_n\}$ converges strongly to an element $w \in \Omega$ and the following inequality holds,
   $$\langle(\gamma f - A)w, p - w\rangle \leq 0, \quad \forall p \in \Omega.$$

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APPENDIX

RESEARCH PUBLICATIONS

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