A COMPARISON OF THREE NUMERICAL METHODS FOR SOLVING THE TWO-DIMENSIONAL UNSTEADY ADVECTION-DIFFUSION EQUATION

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF SCIENCE

MAJOR IN MATHEMATICS

FACULTY OF SCIENCE

UBON RATCHATHANI UNIVERSITY

YEAR 2012

THESIS APPROVAL

UBON RATCHATHANI UNIVERSITY

MASTER OF SCIENCE

MAJOR IN MATHEMATICS FACULTY OF SCIENCE

TITLE A COMPARISON OF THREE NUMERICAL METHODS FOR SOLVING THE TWO-DIMENSIONAL UNSTEADY ADVECTION-DIFFUSION EQUATION

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ACKNOWLEDGEMENTS

I have many people to thank for their help and encouragement on this thesis. First on the list is my thesis advisor, Dr.Sasitorn Pusjuso whose help, stimulating suggestions and encouragement helped me in all the time of research and writing of this thesis. I am also very thankful for her patience in reading the thesis and for making many extensive and valuable comments throughout the entire thesis. I am deeply indebted to my Co-advisor, Dr.Pairin Suwannasri and Dr.Supot Seebut, for their valuable suggestions which so much kindness. I would like to thanks my thesis committees, Dr.Montri Thongmoon and Asst.Prof.Dr.Hathaikarn Wattanataweekul, for their valuable comments and offering suggestions for improvement, especially the strong physical ideas.

I would like to thank the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand for the scholarship to study at Ubon Ratchathani University. I would like to thank the Department of Mathematics, Statistics and Computer, Faculty of Science, for permission me study at Ubon Ratchathani University. I would like to thank Acting. Capt. Dr.Panat Guayjarernpanishk for thesis's template design in LaTex program. I have
furthermore to thank my friends at Department of Mathematics, Statistics and Computer for many helps.

Finally thank to my father and my mother who encouraged me and for their patience during my study. They always support me with all their love.

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ชื่อเรื่อง : การเปรียบเทียบระเบียบวิธีเชิงตัวเลขสามรูปแบบสำหรับการแก้สมการการพาและสมการการแพร่ในปริภูมิสองมิติ

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ศัพท์สำคัญ : สมการการพาและสมการการแพร่ ระเบียบวิธีผลต่างอันตะ ระเบียบวิธีชิ้นประกอบอันตะ

วิทยานิพนธ์นี้นำเสนอการเปรียบเทียบระเบียบวิธีเชิงตัวเลขสามรูปแบบ ได้แก่ ระเบียบวิธีผลต่างอันตะ ระเบียบวิธีชิ้นประกอบอันตะ และระเบียบวิธี operator splitting สำหรับสมการการพาและสมการการแพร่ในปริภูมิสองมิติสมการควบคุมจะถูกแยกออกเป็นสมการการพาและสมการการแพร่ ซึ่งถูกแก้โดยวิธีผลต่างอันตะและวิธีชิ้นประกอบอันตะตามลำดับ การตรวจสอบความถูกต้องของระเบียบวิธีเชิงตัวเลขดังกล่าวดำเนินการโดยเปรียบเทียบผลเฉลยเชิงตัวเลขกับข้อมูลจากห้องปฏิบัติการจำลองทางฟิสิกส์และผลเฉลยเชิงตัวเลขอื่นๆ ที่หาได้สุดท้ายเปรียบเทียบประสิทธิภาพและความถูกต้องของระเบียบวิธีเชิงตัวเลขทั้งสามรูปแบบ

ABSTRACT

TITLE : A COMPARISON OF THREE NUMERICAL METHODS FOR SOLVING THE TWO-DIMENSIONAL UNSTEADY ADVECTION-DIFFUSION EQUATIONS

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DEGREE : MASTER OF SCIENCE

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KEYWORDS : ADVECTION-DIFFUSION EQUATIONS / FINITE DIFFERENCE METHOD/ FINITE ELEMENT METHOD

In this research, a comparative study on three numerical methods, the semi-discrete finite difference method, the Galerkin finite element method and the operator-splitting method for two-dimensional unsteady advection-diffusion problem is presented. The governing equation is splitted into advection and diffusion equations and solved by finite difference method and finite element method, respectively. The numerical algorithm has been validated by comparing with data from laboratory physical models and orther numerical results.

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LIST OF SYMBOLS

SYMBOL	
D	Diffusion coefficient
F	Physical flux in $x-$ direction
G	Physical flux in $y-$ direction
t	Time
α	Velocity component in $x-$ direction
β	Velocity component in $y-$ direction
x	$x-$coordinate
y	$y-$coordinate
ω	Shape function
Ω	Domain
N	Set of counting number
R	Set of real number
R^2	Set of order pair of real number
Δx	mesh width in $x-$ direction
Δy	mesh width in $y-$ direction
Δt	Time increment

LIST OF TECHNICAL TERMS AND ABBREVIATIONS

FDM	Finite Difference Method
FEM	Finite Element Method
OSM	Operator Splitting Method

CHAPTER I

INTRODUCTION

In this chapter, literature review for numerical methods, the semi-discrete central finite difference method, the finite element method, and the operator splitting method, would be firstly presented. And then the objective, scope, and plan of the thesis is described respectively. Finally, it would be ended with expected result from the study.

1.1 Literature review

The models of transport problem involving advective and diffusive arise in many important applications in science and engineering such as heat and mass transfer, play a vitally important role in human life. Gases and liquids surround us, flow inside our bodies, and have a profound influence on the environment in which we live. Fluid flows produce winds, rains, floods, and hurricanes. Convection and diffusion are re-sponsible for temperature fluctuations and transport of pollutants in air, water or soil. The ability to understand, predict, and control transport phenomena is essential for many industrial applications, such as aerodynamic shape design, oil recovery from an underground reservoir, or multiphase/multicomponent flows in furnaces, heat exchangers, and chemical reactors. This ability offers substantial economic benefits and contributes to human well-being. Heating, air conditioning, and weather fore- cast have become an integral part of our everyday life. We take such things for granted and hardly ever think about the physics and mathematics behind them. Various numerical method have been proposed to analyze such problem. There are many numerical methods for to solve the problems such as the finite difference method, the finite element method and the finite volume method. Researchers have developed and simulators for use in the planning and design of numerical method for to solving in these problems.

In this section, we proposed the literature review of the three numerical techniques, the semi-discrete central finite difference method, the finite element method and the operator splitting method, respectively.

1.1.1 The semi-discrete central finite difference method

Many of the recently developed high-resolution scheme for hyperbolic conservation laws are based on differencing. These scheme is the averaging of an approximate Godonov solver and Riemann solver. Riemann solver makes it difficulties for these problems. Therefore, many researchers have tried to invent a new numerical scheme to avoid using the Riemann solver.

H. Nessyahu and E. Tadmor (1990) developed high-resolution schemes for hyperbolic conservation law base on central difference scheme in [11]. They proposed a family of non-oscillatory, second order, central difference and first-order Lax-Friedrichs (LxF) scheme. The main advantage is simplicity: no Riemann problems and using high-resolution MUSCL-type interpolants. R. Kupferman and E. Tadmor (1997) proposed a high-resolution, second-order central differrence method for incompressible flows. The method is based on a recent second-order extension of the class Lax-Friedrichs scheme introduced for hyperbolic conservation laws (H. nessyahu and E. Tadmor (1990)). The scheme is fast, easy to implement, and readily generalizable. The advantage of central scheme, proposed in its velocity formulation is 2-fold : Generalization to the three-dimensional case is straightforward, and the treatment of the boundary condition associated with general geometries becomes simpler. The result is simple, fast high-resolution method, whose accuracy is comparable to that of an upwind scheme. X. D. Liu and E. Tadmor (1998) presented a third -order, non-oscillatory central difference scheme for the approximate solution of nonlinear system of hyperbolic conservation laws in [9]. The third-order central scheme are an extension along the line of the second-order central scheme of Nessyuhu and Tadmor[NT]. The third-order scheme have the advantage of the central scheme over the upwind ones: in that no Riemann solver are involved. The use of the third -order picewise quadratic approximation compensates for the excessive viscosity. The result is a simple, robust, Riemann- solver-free central difference scheme with third-order resolution. F. Bianco, G. Puppo and G. Russo (1999) they presented third and fourth order central schemes for the approximate solution of quasilinear system of conservation law propose in [1]. The schemes are an extension of the second order Nessyahu-Tadmor scheme.

The new second-order semi-discrete schemes applied to the one-and two-dimensional hyperbolic conservation law and convection-diffusion equation is presented by

A. Kurganov and E. Tadmor (2000). The new scheme can be viewed as modifications of the Nessyuhu-Tadmor scheme[11]. This enjoy the major advantage of the central scheme over the upwind ones : first, no Riemann solver are involved, and second as the result of the being Riemann-solver-free their realization and generalization for complicated multidimensional. This scheme have the smaller amount of numerical viscosity than the original NT scheme, and unlike other central scheme, the can be written and efficiently intergrated in their semi-discrete form.

1.1.2 The finite element method

The finite element method originated from the need for solving complex elasticity and structural analysis problems in civil and aeronautical engineering. Its development can be traced back to the work by Alexander Hrennikoff (1941) and Richard Courant (1942). While the approaches used by these pioneers are different, they share one essential characteristic: mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements. Starting in 1947, Olgierd Zienkiewicz from Imperial College gathered those methods together into what would be called the Finite Element Method, building the pioneering mathematical formalism of the method.

Development of the finite element method began in earnest in the middle to late 1950s for airframe and structural analysis and gathered momentum at the University of Stuttgart through the work of John Argyris and at Berkeley through the work of Ray W. Clough in the 1960s for use in civil engineering. By late 1950s, the key concepts of stiffness matrix and element assembly existed essentially in the form used today. NASA issued a request for proposals for the development of the finite element software NASTRAN in 1965. The method was again provided with a rigorous mathematical foundation in 1973 with the publication of Strang and Fix's An Analysis of The Finite Element Method, and has since been generalized into a branch of applied mathematics for numerical modeling of physical systems in a wide variety of engineering disciplines, e.g., electromagnetism and fluid dynamics. Many researchers have developed and simulators for use in the planning and design of the numerical methods for the solving in these problems.

A. Mizukami and T. J. R. Hughes (1985) presented a new Petrov-Galerkin method for convection-diffusion flow problem which is conservative and satisfies the discrete maximum principle. this method possesses no spurious crosswind diffusion and gives vary accurate solution. The methods is representative of a class of methods which may be described as 'fixed-mesh adaptive'.

M. Feistauer and V. Kučera (2008) analysis of the discontinuous Galerkin finite element method (DGFEM) for the numerical solution of nonstationary nonlinear convection-diffusion problems equipped with Dirichlet boundary conditions.

T. Sun (2010) proposed a discontinuous Galerkin finite element method with interior penalties for convection-diffusion optimal control problem. A semi-discrete time DG scheme for this problem is presented. In this work analyze the stability of this scheme, and derive a priori and a posteriori error estimates for both the state and the control approximation.

Recently, V. John and J. Novo (2011) proposed the conditions on the stabilization parameters are explored for different approaches in deriving error estimates for the SUPG finite element stabilization of time-dependent convection diffusion-reaction equations that is combined with the backward Euler method. For this reason, the time-continuous case is analyzed under certain conditions on the coefficients of the equation and the finite element method. An error estimate with the standard order of convergence is derived for stabilization parameters of the same form that is optimal for the steady-state problem.

1.1.3 The operator splitting method

Operator splitting is a powerful method for numerical investigation of complex models. The basic idea of the operator splitting methods based on splitting of complex problem into a sequence of simpler tasks, called split sub-problems. The sub operators are usually chosen with regard to different physical process. Then instead of the original problem, a sequence of sub models are solved, which gives rise to a splitting error. The order of the splitting error can be estimate theoretically. In practice, splitting procedures are associated with different numerical methods for solving the sub-problems, which also causes a certain amount of error.

The idea of operator splitting, which was the Lie-Trotter splitting, dates back to the 1950s. It was probably in 1957 that this method was first used in the solution of partial differential equations (Bagrinovskii & Godunov, 1957). The first splitting methods were developed in the 1960s or 1970s and were based on fundamental results of finite difference methods. The classical splitting methods are the Lie-Trotter splitting, the Strang splitting (Dimov et al., 2001), (Strang, 1968), (Faragó & Havasi, 2007) and the symmetrically weighted splitting (Strang, 1963), (Csomós et al., 2005). A renewal of the methods was done. In the 1980s while using the methods or complex process underlying partial differential methods in (Crandall & Majda, 1999).

Complex physical processes are frequently modelled by the systems of linear or non-linear partial differential equations. Due to the complexity of these equations, typically there is no numerical method which can provide a numerical solution that is accurate enough while taking reasonable integrational time. In order to simplify the task (Strang, 1968), (Marchuk, 1988) operator splitting procedure has been introduced, which is widely used for solving advection-diffusion-reaction problems in (Hvistendahl et al., 2001), (Marinova et al., 2003) Navier-Stokes equation in (Christov & Marinova, 2001), including modelling turbulence (Mimura et al., 1984) and interfaces.

In many applications in the past, a mixing of various terms in the equations for the discretization and solver methods made it difficult to solve them together. With respect to the adapted methods for a simpler equation, the methods give improved results for simpler part. The higher order operator splitting methods are used for more accurate computations, but also with respect to more computational steps. There have been some composition techniques to get the higher order splitting methods. The well known higher order composition schemes are developed by many authers (Blanes and Moan, 2002), (Kahan & Li, 1997), (Mclachlan & Quispel, 2002), (Suzuki, 1990), (Yoshida, 1990).

The idea behind this type of approach is that the overall evolution operator is formally written as a sum of evolution operators for each term (operator) in the model. In other words, one splits the model into a set of sub-equations, where each sub-equation is of a type for which simpler and more practical algorithms are available. The overall numerical method is then formed by picking an appropriate numerical scheme for each sub-equation and piecing the schemes together by operator splitting.

A. Chertock, A. kurganov and G. petrova (2009) propose a second-order fast explicit operator splitting (FEOS) method based on the strang splitting. The main idea of the method is to solve palabolic problem via a discretization of the fomular for the exact solution of the heat equation, which is realized using a conservative and accurate quandrature formula. The hyperbolic problem is solved by a second-order finite-volume Godunov-type scheme.

A. Chertock, C.R. Doering, E. Kashdan and A. Kurganov (2010) proposed splitting approach, the hyperbolic and parabolic subproblems have been solved by two different methods: The hyperbolic equation has been solved numerically by the second-order Godunov-type central-upwind scheme in [2], while the parabolic equation has been solved exactly using the pseudo-spectral technique.

1.2 Objectives

(i) To derive numerical schemes based on finite difference method, finite element method and operator splitting method for two-dimensional unsteady advection-diffusion equation

(ii) To verify numerical algorithm by comparing with data from laboratory physical models which are known exact or approximate solution

(iii) To compare the efficiency of the finite difference method, the finite element method, and the operator splitting method

1.3 Scope of the thesis

This thesis investigates the study of comparative on three numerical methods, the semi-discrete finite difference method, the Galerkin finite element method, and the operator-splitting method for two-dimensional unsteady advection-diffusion equations on rectangular domain.

1.4 Plan of the thesis

This thesis, we study of comparative on three numerical methods for solving the two-dimensional unsteady advection-diffusion equations. The first part gives the detail of the literature review of the finite semi-discrete central difference method, the Galerkin finite element method, and the operator splitting method.

The second part gives the computation of the numerical scheme of the finite difference method, the finite element method, and the operator splitting method for solving the two-dimensional unsteady advection-diffusion equations.

The third part, verify numerical algorithm by comparing with data from laboratory physical models which are known exact or approximate solution is presented.

The fourth part we show that the numerical experiment obtained by a comparison of the finite difference method, the finite element method, and the operator splitting method(OSM), respectively. All of the computer programs which appears in this thesis are coded from their numerical methods. These are developed and constructed by using MATLAB programming.

The fifth part we devoted to a brief conclusion. Finally some references are introduced at the end.

1.5 Expected results

(i) The numerical schemes are analized to create procedure for programming a computer to solve the two-dimensional unsteady advection-diffusion equation.

(ii) The result from the study could be applied to any area based on finite difference and finite element method for method for two-dimensional unsteady advection-diffusion equation such as water pollution, air pollution and so on.

CHAPTER II

MATHEMATICAL MODELS

In this chapter, we described the advection equation, the diffusion equation and the advection-diffusion equations. The mathematical model for the one-and two-dimensional unsteady advection-diffusion equations is presented.

2.1 The Advection Equation

Advection, in chemistry, engineering and earth sciences, is a transport mechanism of a substance, or a conserved property, by a fluid, due to the fluid's bulk motion in a particular direction. An example of advection is the transport of pollutants or silt in a river. The motion of the water carries these impurities downstream. Another commonly advected property is energy or enthalpy, and here the fluid may be water, air, or any other thermal energy-containing fluid material. Any substance, or conserved property (such as enthalpy) can be advected, in a similar way, in any fluid. The fluid motion in advection is described mathematically as a vector field, and the material transported is typically described as a scalar concentration of substance, which is contained in the fluid. Advection requires currents in the fluid, and so cannot happen in rigid solids. It does not include transport of substances by simple diffusion.

Occasionally, the term advection is used as synonymous with convection. However, many engineers prefer to use the term convection to describe transport by combined molecular and eddy diffusion, and reserve the usage of the term advection to describe transport with a general (net) flow of the fluid (like in river or pipeline).

The advection equation is the partial differential equation that governs the motion of a conserved scalar as it is advected by a known velocity field. It is derived using the scalar's conservation law, together with Gauss's theorem, and taking the infinitesimal limit.

The one-dimensional unsteady advection equation is the following form as

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0 \tag{2.1}$$

where

$u(x,t)$: concentration averaged in depth,

α : the velocity in X direction,

The two-dimensional unsteady advection equation is the following form as

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} = 0 \tag{2.2}$$

where

$u(x,y,t)$: concentration averaged in depth at the point (x,y),

α : the velocity in X direction,

β : the velocity in Y direction.

2.2 The Diffusion Equation

A fundamental transport process in environmental fluid mechanics is diffusion. Diffusion differs from advection in that it is random in nature (does not necessarily follow a fluid particle). A well-known example is the diffusion of perfume in an empty room. If a bottle of perfume is opened and allowed to evaporate into the air, soon the whole room will be scented. We know also from experience that the scent will be stronger near the source and weaker as we move away, but fragrance molecules will have wondered throughout the room due to random molecular and turbulent motions. Thus, diffusion has two primary properties: it is random in nature, and transport is from regions of high concentration to low concentration, with an equilibrium state of uniform concentration.

Next, we introduce to the one-and two-dimensional unsteady diffusion equation.

The one-dimensional unsteady diffusion equation is the following form as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{2.3}$$

where

$u(x,t)$: concentration averaged in depth,

D : diffusion coefficient in X direction.

The two-dimensional unsteady diffusion equation is the following form as

$$\frac{\partial u}{\partial t} = D(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{2.4}$$

where

$u(x, y, t)$: concentration at the point (x, y) in Ω,

D : diffusion coefficients in X, Y directions.

2.3 The Advection-Diffusion Equation

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection. From the definition above it follows that the convection–diffusion equation combines both parabolic and hyperbolic partial differential equations.

A mathematical model for described the dispersion of the concentration in one-and two-dimensional domain. In this case, the dispersion of the concentration is described by the unsteady advection-diffusion equation with constant coefficients in one and two-dimensional domain $\Omega \in R^2$,

The one-dimensional unsteady advection-diffusion equations is the following form as

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} \tag{2.5}$$

where

$u(x, t)$: concentration averaged in depth,

α : the velocity in X direction,

D : diffusion coefficient in X direction.

The two-dimensional unsteady advection-diffusion equations is the following form as

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.6}$$

where

$u(x, y, t)$: concentration at the point (x, y) in Ω,

α : the velocity in X direction,

β : the velocity in Y direction,

D : diffusion coefficients in X, Y directions.

CHAPTER III

NUMERICAL METHODS

In this chapter, we consider the semi-discrete central finite difference method, the Galerkin finite element method, and the operator splitting method for two-dimensional unsteady advection-diffusion problems.

3.1 The Semi-Discrete Central Finite Difference Method

We will consider a uniform grids and use the following natation : $(x_i, y_j) = (i\Delta x, j\Delta y)$, $(x_{i\pm\frac{1}{2}}, y_{j\pm\frac{1}{2}}) = ((i \pm \frac{1}{2})\Delta x,\ (j \pm \frac{1}{2})\Delta y)$, $t^n = n\Delta t$, $u^n_{i,j} = u(x_i, y_j, t^n)$, where $\Delta x, \Delta y$ and Δt are small spatial and time scales, respectively(see as Figure 3.1).

According to [13], the semi-discrete central scheme for

$$\begin{aligned}
\frac{d}{dt}u_{i,j}(t) &= -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2},j}(t) - F_{i-\frac{1}{2},j}(t)\right) - \frac{1}{\Delta y}\left(G_{i,j+\frac{1}{2}}(t) - G_{i,j-\frac{1}{2}}(t)\right) \\
&\quad + \frac{1}{\Delta x^2}\left(u_{i+1,j}(t) - 2u_{i,j}(t) + u_{i-1,j}(t)\right) \\
&\quad + \frac{1}{\Delta y^2}\left(u_{i,j+1}(t) - 2u_{i,j}(t) + u_{i,j-1}(t)\right) \qquad (3.1)
\end{aligned}$$

where $F_{i+\frac{1}{2},j}(t)$ and $G_{i,j+\frac{1}{2}}(t)$ are x and y numerical advection fluxs, respectively,

$$F_{i+\frac{1}{2},j}(t) = \frac{\alpha u^+_{i+\frac{1}{2},j}(t) + \alpha u^-_{i+\frac{1}{2},j}(t)}{2} - \frac{a^x_{i+\frac{1}{2},j}(t)}{2}\left[u^+_{i+\frac{1}{2},j}(t) - u^-_{i+\frac{1}{2},j}(t)\right] \qquad (3.2)$$

$$G_{i,j+\frac{1}{2}}(t) = \frac{\beta u^+_{i,j+\frac{1}{2}}(t) + \beta u^-_{i,j+\frac{1}{2}}(t)}{2} - \frac{a^y_{i,j+\frac{1}{2}}(t)}{2}\left[u^+_{i,j+\frac{1}{2}}(t) - u^-_{i,j+\frac{1}{2}}(t)\right] \qquad (3.3)$$

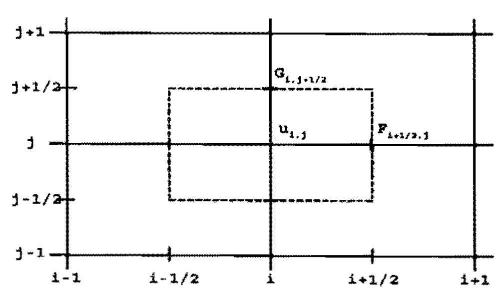


Figure 3.1 : Spatial grid FDM

which are expressed in term of the intermediate values

$$
\begin{aligned}
u^-_{i+\frac{1}{2},j} &= u_{i,j}(t) + \frac{\Delta x}{2}\varphi^x_{i,j}(t) && (3.4)\\
u^+_{i+\frac{1}{2},j} &= u_{i+1,j}(t) - \frac{\Delta x}{2}\varphi^x_{i+1,j}(t) && (3.5)\\
u^-_{i,j+\frac{1}{2}} &= u_{i,j}(t) + \frac{\Delta y}{2}\varphi^y_{i,j}(t) && (3.6)\\
u^+_{i,j+\frac{1}{2}} &= u_{i,j+1}(t) - \frac{\Delta y}{2}\varphi^y_{i,j+1}(t) && (3.7)
\end{aligned}
$$

where the limiters are defined by

$$
\begin{aligned}
\varphi^x_{i,j}(t) &= \text{minmod}\left(\frac{u_{i,j}(t) - u_{i-1,j}(t)}{\Delta x}, \frac{u_{i+1,j}(t) - u_{i,j}(t)}{\Delta x}\right) && (3.8)\\
\varphi^y_{i,j}(t) &= \text{minmod}\left(\frac{u_{i,j}(t) - u_{i,j-1}(t)}{\Delta y}, \frac{u_{i,j+1}(t) - u_{i,j}(t)}{\Delta y}\right) && (3.9)
\end{aligned}
$$

and the local speeds $a^x_{i+\frac{1}{2},j}(t)$ and $a^y_{i,j+\frac{1}{2}}(t)$, are computed by

$$
\begin{aligned}
a^x_{i+\frac{1}{2},j}(t) &= \max(|\alpha|) && (3.10)\\
a^y_{i,j+\frac{1}{2}}(t) &= \max(|\beta|) && (3.11)
\end{aligned}
$$

where the minmod limiter of two arguments is defined by

$$
\text{minmod}(a, b) = \begin{cases} a, & \text{if} \quad |a| \le |b| \quad \text{and} \quad ab > 0\ ; \\ b, & \text{if} \quad |b| \le |a| \quad \text{and} \quad ab > 0\ ; \\ 0, & \text{if} \quad ab < 0. \end{cases} \tag{3.12}
$$

The time discretization will be implemented by the class of third order TVD Runge-kutta methods.

$$
\frac{du}{dt} = L(u) \tag{3.13}
$$

$$
\begin{aligned}
u^1 &= u^n + \Delta t L(u^n) && (3.14)\\
u^2 &= \frac{3}{4}u^n + \frac{1}{4}u^1 + \frac{1}{4}\Delta t L(u^1) && (3.15)\\
u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^2 + \frac{2}{3}\Delta t L(u^2) && (3.16)
\end{aligned}
$$

3.2 The Galerkin Finite Element Method

Applying the method of weight residual to Eq.(2.6) and integrating of the differential equation and boundary condition is

$$\int_{\Omega} \omega \frac{\partial u}{\partial t}\, d\Omega + \int_{\Omega} \omega \left(\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} \right) d\Omega - \int_{\Omega} D\omega \left(\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} \right) d\Omega. \tag{3.17}$$

We want evaluate the first term of diffusion equation (3.17)

$$\int_{\Omega} D\omega \left(\frac{\partial u^2}{\partial x^2} \right) d\Omega. \tag{3.18}$$

The domain integral can be expressed as

$$\int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} D\omega \frac{\partial u^2}{\partial x^2}\, dx \right) dy, \tag{3.19}$$

where y_1 and y_2 are the minimum and maximum value of the domain of the y-axis as the strip along the x-axis move in the y-direction. Integrating by part with respect to x yields.

$$\int_{x_1}^{x_2} D\omega \frac{\partial u^2}{\partial x^2}\, dx = \left[D\omega \frac{\partial u^2}{\partial x^2} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} D \left(\frac{\partial u}{\partial x} \frac{\partial \omega}{\partial x} \right) dx. \tag{3.20}$$

Substitute Eq.(3.20) into (3.18), we obtain

$$\int_{y_1}^{y_2} \left[D\omega \frac{\partial u^2}{\partial x^2} \right]_{x_1}^{x_2} dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} D \left(\frac{\partial u}{\partial x} \frac{\partial \omega}{\partial x} \right) dx\, dy \tag{3.21}$$

and rewriting the expression using the domain and boundary integrations

$$\int_{\Gamma_2} D\omega \frac{\partial u}{\partial x} \eta_x\, d\Gamma - \int_{\Gamma_1} D\omega \frac{\partial u}{\partial x} \eta_x\, d\Gamma - \int_{\Omega} D \left(\frac{\partial u}{\partial x} \frac{\partial \omega}{\partial x} \right) d\Omega. \tag{3.22}$$

in which η_x is the x-component of the normal vector which is assumed to be positive in the outward direction. Finally combining the two-boundary integrals gives

$$-\int_\Omega D\left(\frac{\partial u}{\partial x}\frac{\partial \omega}{\partial x}\right) d\Omega + \oint_\Gamma D\omega\frac{\partial u}{\partial x}\eta_x \, d\Gamma. \tag{3.23}$$

Similarly, the second term of diffusion in Eq.(3.17) can be written as

$$-\int_\Omega D\left(\frac{\partial u}{\partial y}\frac{\partial \omega}{\partial y}\right) d\Omega + \oint_\Gamma D\omega\frac{\partial u}{\partial y}\eta_y \, d\Gamma. \tag{3.24}$$

Adding Eqs.(3.23) and (3.24) produces

$$\int_\Omega D\omega(\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2})\, d\Omega = \int_\Omega D\left(\frac{\partial u}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \omega}{\partial y}\right) d\Omega + \oint_\Gamma D\omega\left(\frac{\partial u}{\partial x}\eta_x + \frac{\partial u}{\partial y}\eta_y\right) d\Gamma. \tag{3.25}$$

Since the boundary integral can be written as

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\eta_x + \frac{\partial u}{\partial y}\eta_y \tag{3.26}$$

we can rewrite Eq.(3.25) as

$$\int_\Omega D\omega\left(\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2}\right) d\Omega = -\int_\Omega D\left(\frac{\partial u}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \omega}{\partial y}\right) d\Omega + \oint_\Gamma D\omega\frac{\partial u}{\partial n}\, d\Gamma. \tag{3.27}$$

The symbol $\oint$ to denote the line integral around a closed boundary is replaced by $\int$ for simplicity in the following text. Then, Equation (3.27) can be written as

$$\int_\Omega \omega\frac{\partial u}{\partial t}\, d\Omega + \int_\Omega D\left(\frac{\partial u}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \omega}{\partial y}\right) d\Omega = \oint_{\Gamma_n} D\omega\frac{\partial u}{\partial n}\, d\Gamma. \tag{3.28}$$

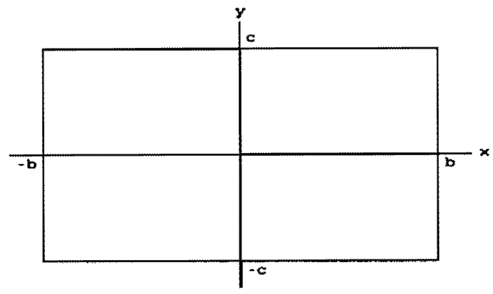


Figure 3.2 : Bilinear rectangular element

Bilinear Rectangular Element

The bilinear rectangular element is show in Fig. 3.2. The shape function for this element can be derived from the following interpolation function

$$u = a_1 + a_2x + a_3y + a_4xy \tag{3.29}$$

For the finite element computation, the element nodal sequence must be in the same direction for every element in the domain.

By the Galerkin method, thus approximate unknown nodal using the price-wise approximation

$$u(x,y,t) = \sum_{i=1}^{4} u_i(t)H_i(x,y) \tag{3.30}$$

where a linear basis or shape function is used in this study and they are defined as,

$$H_1(x,y) = \frac{1}{4bc}(b-x)(c-y) \tag{3.31}$$
$$H_2(x,y) = \frac{1}{4bc}(b+x)(c-y) \tag{3.32}$$
$$H_3(x,y) = \frac{1}{4bc}(b+x)(c+y) \tag{3.33}$$
$$H_4(x,y) = \frac{1}{4bc}(b-x)(c-y) \tag{3.34}$$

where $2b$ and $2c$ are length and height of the element, respectively.

Discretizing the whole into these non-overlapping finite element and using the shape function of these element as nodal point, we get the final assembled form as

$$M\{\dot{u}\}^t + S\{u\}^t + K\{u\}^t = \{F\}^t \tag{3.35}$$

where square matrices of order for all $i = 1, 2, 3, ..., n$ which are

$$M_{i,j} = \int_e H_i H_j \, d\Omega, \tag{3.36}$$

$$S_{i,j} = \int_e \Big(\alpha \frac{\partial H_i}{\partial x} + \beta \frac{\partial H_i}{\partial y}\Big) H_j \, d\Omega, \tag{3.37}$$

$$K_{i,j} = \int_e \Big(\frac{\partial H_i}{\partial x}\frac{\partial H_j}{\partial x} + \frac{\partial H_i}{\partial y}\frac{\partial H_j}{\partial y}\Big) \, d\Omega, \tag{3.38}$$

$$F_i = \int_e H_i \, d\Omega. \tag{3.39}$$

where M, S and K are the 4x4 element matrix,

$$\begin{aligned} M_{i,j} &= \int_e (H_i H_j) \, d\Omega \\ &= \int_e \left(\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{Bmatrix} \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} \right) d\Omega \end{aligned} \tag{3.40}$$

where

$$M_{11} = \int_e H_1 H_1 \, d\Omega, \qquad M_{31} = \int_e H_3 H_1 \, d\Omega, \tag{3.41}$$

$$M_{12} = \int_e H_1 H_2 \, d\Omega, \qquad M_{32} = \int_e H_3 H_2 \, d\Omega, \tag{3.42}$$

$$M_{13} = \int_e H_1 H_3 \, d\Omega, \qquad M_{33} = \int_e H_3 H_3 \, d\Omega, \tag{3.43}$$

$$M_{14} = \int_e H_1 H_4 \, d\Omega, \qquad M_{34} = \int_e H_3 H_4 \, d\Omega, \tag{3.44}$$

$$M_{21} = \int_e H_2 H_1 \, d\Omega, \qquad M_{41} = \int_e H_4 H_1 \, d\Omega, \tag{3.45}$$

$$M_{22} = \int_e H_2 H_2 \, d\Omega, \qquad M_{42} = \int_e H_4 H_2 \, d\Omega, \tag{3.46}$$

$$M_{23} = \int_e H_2 H_3 \, d\Omega, \qquad M_{43} = \int_e H_4 H_3 \, d\Omega, \tag{3.47}$$

$$M_{24} = \int_e H_2 H_4 \, d\Omega, \qquad M_{44} = \int_e H_4 H_4 \, d\Omega, \tag{3.48}$$

in which

$$
\begin{aligned}
S_{i,j} &= \int_e \left(\alpha \frac{\partial Hi}{\partial x} + \beta \frac{\partial Hi}{\partial y} \right) H_j \, d\Omega \\
&= \int_e \left(\alpha \begin{Bmatrix} \frac{\partial H1}{\partial x} \\ \frac{\partial H2}{\partial x} \\ \frac{\partial H3}{\partial x} \\ \frac{\partial H4}{\partial x} \end{Bmatrix} \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} + \beta \begin{Bmatrix} \frac{\partial H1}{\partial y} \\ \frac{\partial H2}{\partial y} \\ \frac{\partial H3}{\partial y} \\ \frac{\partial H4}{\partial y} \end{Bmatrix} \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} \right) d\Omega
\end{aligned} \tag{3.49}
$$

where

$$S_{11} = \int_e \left(\alpha \frac{\partial H1}{\partial x} H_1 + \beta \frac{\partial H1}{\partial y} H_1 \right) \, d\Omega, \tag{3.50}$$

$$S_{12} = \int_e \left(\alpha \frac{\partial H1}{\partial x} H_2 + \beta \frac{\partial H1}{\partial y} H_2 \right) \, d\Omega, \tag{3.51}$$

$$S_{13} = \int_e \left(\alpha \frac{\partial H1}{\partial x} H_3 + \beta \frac{\partial H1}{\partial y} H_3 \right) \, d\Omega, \tag{3.52}$$

$$S_{14} = \int_e \left(\alpha \frac{\partial H1}{\partial x} H_4 + \beta \frac{\partial H1}{\partial y} H_4 \right) \, d\Omega, \tag{3.53}$$

$$S_{21} = \int_e \left(\alpha \frac{\partial H2}{\partial x} H_1 + \beta \frac{\partial H2}{\partial y} H_1 \right) \, d\Omega, \tag{3.54}$$

$$S_{22} = \int_e \left(\alpha \frac{\partial H2}{\partial x} H_2 + \beta \frac{\partial H2}{\partial y} H_2 \right) \, d\Omega, \tag{3.55}$$

$$S_{23} = \int_e \left(\alpha \frac{\partial H2}{\partial x} H_3 + \beta \frac{\partial H2}{\partial y} H_3 \right) \, d\Omega, \tag{3.56}$$

$$S_{24} = \int_e \left(\alpha \frac{\partial H2}{\partial x} H_4 + \beta \frac{\partial H2}{\partial y} H_4 \right) \, d\Omega, \tag{3.57}$$

$$S_{31} = \int_e \left(\alpha \frac{\partial H3}{\partial x} H_1 + \beta \frac{\partial H3}{\partial y} H_1 \right) \, d\Omega, \tag{3.58}$$

$$S_{32} = \int_e \left(\alpha \frac{\partial H3}{\partial x} H_2 + \beta \frac{\partial H3}{\partial y} H_2 \right) \, d\Omega, \tag{3.59}$$

$$S_{33} = \int_e \left(\alpha \frac{\partial H3}{\partial x} H_3 + \beta \frac{\partial H3}{\partial y} H_3 \right) \, d\Omega, \tag{3.60}$$

$$S_{34} = \int_e \left(\alpha \frac{\partial H3}{\partial x} H_4 + \beta \frac{\partial H3}{\partial y} H_4 \right) \, d\Omega, \tag{3.61}$$

$$S_{41} = \int_e \left(\alpha \frac{\partial H4}{\partial x} H_1 + \beta \frac{\partial H4}{\partial y} H_1 \right) \, d\Omega, \tag{3.62}$$

$$S_{42} = \int_e \left(\alpha \frac{\partial H4}{\partial x} H_2 + \beta \frac{\partial H4}{\partial y} H_2 \right) \, d\Omega, \tag{3.63}$$

$$S_{43} = \int_e \left(\alpha \frac{\partial H4}{\partial x} H_3 + \beta \frac{\partial H4}{\partial y} H_3 \right) d\Omega, \tag{3.64}$$

$$S_{44} = \int_e \left(\alpha \frac{\partial H4}{\partial x} H_4 + \beta \frac{\partial H4}{\partial y} H_4 \right) d\Omega, \tag{3.65}$$

in which

$$\begin{aligned} K_{i,j} &= \int_e \left(\frac{\partial Hi}{\partial x} \frac{\partial Hj}{\partial x} + \frac{\partial Hi}{\partial y} \frac{\partial Hj}{\partial y} \right) d\Omega \\ &= \int_e \left(\begin{Bmatrix} \frac{\partial H1}{\partial x} \\ \frac{\partial H2}{\partial x} \\ \frac{\partial H3}{\partial x} \\ \frac{\partial H4}{\partial x} \end{Bmatrix} \begin{bmatrix} \frac{\partial H1}{\partial x} & \frac{\partial H2}{\partial x} & \frac{\partial H3}{\partial x} & \frac{\partial H4}{\partial x} \end{bmatrix} \right. \\ &\quad \left. + \begin{Bmatrix} \frac{\partial H1}{\partial y} \\ \frac{\partial H2}{\partial y} \\ \frac{\partial H3}{\partial y} \\ \frac{\partial H4}{\partial y} \end{Bmatrix} \begin{bmatrix} \frac{\partial H1}{\partial y} & \frac{\partial H2}{\partial y} & \frac{\partial H3}{\partial y} & \frac{\partial H4}{\partial y} \end{bmatrix} \right) d\Omega \end{aligned} \tag{3.66}$$

where

$$K_{11} = \int_e \left(\frac{\partial H1}{\partial x} \frac{\partial H1}{\partial x} + \frac{\partial H1}{\partial y} \frac{\partial H1}{\partial y} \right) d\Omega, \tag{3.67}$$

$$K_{12} = \int_e \left(\frac{\partial H1}{\partial x} \frac{\partial H2}{\partial x} + \frac{\partial H1}{\partial y} \frac{\partial H2}{\partial y} \right) d\Omega, \tag{3.68}$$

$$K_{13} = \int_e \left(\frac{\partial H1}{\partial x} \frac{\partial H3}{\partial x} + \frac{\partial H1}{\partial y} \frac{\partial H3}{\partial y} \right) d\Omega, \tag{3.69}$$

$$K_{14} = \int_e \left(\frac{\partial H1}{\partial x} \frac{\partial H4}{\partial x} + \frac{\partial H1}{\partial y} \frac{\partial H4}{\partial y} \right) d\Omega, \tag{3.70}$$

$$K_{21} = \int_e \left(\frac{\partial H2}{\partial x} \frac{\partial H1}{\partial x} + \frac{\partial H2}{\partial y} \frac{\partial H1}{\partial y} \right) d\Omega, \tag{3.71}$$

$$K_{22} = \int_e \left(\frac{\partial H2}{\partial x} \frac{\partial H2}{\partial x} + \frac{\partial H2}{\partial y} \frac{\partial H2}{\partial y} \right) d\Omega, \tag{3.72}$$

$$K_{23} = \int_e \left(\frac{\partial H2}{\partial x} \frac{\partial H3}{\partial x} + \frac{\partial H2}{\partial y} \frac{\partial H3}{\partial y} \right) d\Omega, \tag{3.73}$$

$$K_{24} = \int_e \left(\frac{\partial H2}{\partial x} \frac{\partial H4}{\partial x} + \frac{\partial H2}{\partial y} \frac{\partial H4}{\partial y} \right) d\Omega, \tag{3.74}$$

$$
\begin{aligned}
K_{31} &= \int_e \left(\frac{\partial H3}{\partial x} \frac{\partial H1}{\partial x} + \frac{\partial H3}{\partial y} \frac{\partial H1}{\partial y} \right) d\Omega, && (3.75)\\
K_{32} &= \int_e \left(\frac{\partial H3}{\partial x} \frac{\partial H2}{\partial x} + \frac{\partial H3}{\partial y} \frac{\partial H2}{\partial y} \right) d\Omega, && (3.76)\\
K_{33} &= \int_e \left(\frac{\partial H3}{\partial x} \frac{\partial H3}{\partial x} + \frac{\partial H3}{\partial y} \frac{\partial H3}{\partial y} \right) d\Omega, && (3.77)\\
K_{34} &= \int_e \left(\frac{\partial H3}{\partial x} \frac{\partial H4}{\partial x} + \frac{\partial H3}{\partial y} \frac{\partial H4}{\partial y} \right) d\Omega, && (3.78)\\
K_{41} &= \int_e \left(\frac{\partial H4}{\partial x} \frac{\partial H1}{\partial x} + \frac{\partial H4}{\partial y} \frac{\partial H1}{\partial y} \right) d\Omega, && (3.79)\\
K_{42} &= \int_e \left(\frac{\partial H4}{\partial x} \frac{\partial H2}{\partial x} + \frac{\partial H4}{\partial y} \frac{\partial H2}{\partial y} \right) d\Omega, && (3.80)\\
K_{43} &= \int_e \left(\frac{\partial H4}{\partial x} \frac{\partial H3}{\partial x} + \frac{\partial H4}{\partial y} \frac{\partial H3}{\partial y} \right) d\Omega, && (3.81)\\
K_{44} &= \int_e \left(\frac{\partial H4}{\partial x} \frac{\partial H4}{\partial x} + \frac{\partial H4}{\partial y} \frac{\partial H4}{\partial y} \right) d\Omega. && (3.82)
\end{aligned}
$$

Their element matrix for $i^{th}-$ element are

$$
[M^e] = \int_e H_i H_j \, d\Omega = \begin{bmatrix} \frac{4bc}{36} & \frac{2bc}{36} & \frac{bc}{36} & \frac{2bc}{36} \\ \frac{2bc}{36} & \frac{4bc}{36} & \frac{2bc}{36} & \frac{bc}{36} \\ \frac{bc}{36} & \frac{2bc}{36} & \frac{4bc}{36} & \frac{2bc}{36} \\ \frac{2bc}{36} & \frac{bc}{36} & \frac{2bc}{36} & \frac{4bc}{36} \end{bmatrix} \tag{3.83}
$$

$$
[S^e] = \int_e \left(\alpha \frac{\partial H_i}{\partial x} + \beta \frac{\partial H_i}{\partial y} \right) H_j \, d\Omega = \begin{bmatrix} -\frac{(\alpha c+\beta b)}{3} & \frac{\alpha c}{3} - \frac{\beta b}{6} & \frac{(\alpha c+\beta b)}{6} & -\frac{\alpha c}{6} + \frac{\beta b}{3} \\ -\frac{\alpha c}{3} - \frac{\beta b}{6} & \frac{(\alpha c-\beta b)}{3} & \frac{\alpha c}{6} + \frac{\beta b}{3} & -\frac{\alpha c}{6} - \frac{\beta b}{6} \\ -\frac{(\alpha c+\beta b)}{6} & \frac{\alpha c}{6} - \frac{\beta b}{3} & \frac{\alpha c+\beta b}{3} & \frac{\alpha c}{3} + \frac{\beta b}{3} \\ -\frac{\alpha c}{6} - \frac{\beta b}{3} & \frac{(\alpha c-\beta b)}{6} & \frac{\alpha c}{3} + \frac{\beta b}{6} & -\frac{(\alpha c-\beta b)}{3} \end{bmatrix} \tag{3.84}
$$

$$
[K^e] = \int_e \left(\frac{\partial H_i}{\partial x} \frac{\partial H_j}{\partial x} + \frac{\partial H_i}{\partial y} \frac{\partial H_j}{\partial y} \right) d\Omega = \begin{bmatrix} \frac{c^2+b^2}{3bc} & \frac{b^2-2c^2}{6bc} & -\frac{b^2+c^2}{6bc} & \frac{c^2-2b^2}{6bc} \\ \frac{b^2-2c^2}{6bc} & \frac{c^2+b^2}{3bc} & \frac{c^2-2b^2}{6bc} & -\frac{b^2+c^2}{6bc} \\ -\frac{b^2+c^2}{6bc} & \frac{c^2-2b^2}{6bc} & \frac{c^2+b^2}{3bc} & \frac{b^2-2c^2}{6bc} \\ \frac{c^2-2b^2}{6bc} & -\frac{b^2+c^2}{6bc} & \frac{b^2-2c^2}{6bc} & \frac{c^2+b^2}{3bc} \end{bmatrix} \tag{3.85}
$$

Time Integration Technique

In this section, we explain the Crank-Nicolson method for the time derivative. For this technique we write equation (3.35) at time $t + \frac{\Delta t}{2}$ instead of t Then,

$$M\{\dot{u}\}^{t+\frac{\Delta t}{2}} + S\{u\}^{t+\frac{\Delta t}{2}} + K\{u\}^{t=\frac{\Delta t}{2}} = \{F\}^{t+\frac{\Delta t}{2}} \tag{3.86}$$

The time derivative term is expressed using the central difference technique like

$$\{\dot{u}\}^{t+\frac{\Delta t}{2}} = \frac{\{u\}^{t+\Delta t} - \{u\}^{t}}{\Delta t} \tag{3.87}$$

On the other hand, the other term are computed are average like

$$\{u\}^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\{u\}^{t} + \{u\}^{t+\Delta t}\right) \tag{3.88}$$

and

$$\{F\}^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\{F\}^{t} - \{F\}^{t+\Delta t}\right). \tag{3.89}$$

Substitution of Eqs.(3.87) through Eqs.(3.89) into Eqs. (3.86) yields

$$\begin{aligned}(2[M] + \Delta t[S+K])\{u\}^{t+\Delta t} &= \Delta t\left(\{F\}^{t} + \{F\}^{t+\Delta t}\right) - 2[M] \\ &\quad + \Delta t[S+K]\{u\}^{t}. \end{aligned} \tag{3.90}$$

3.3 The Operator-Splitting Method

A second order strang operator splitting method combining the finite difference and finite element method is applied to the two-dimensional advection-diffusion equation. The governing equation is split into the hyperbolic equation and the parabolic equation. The hyperbolic equation is solved by the second order semi-discrete central scheme, and the parabolic equation is solved by the finite element method.

The two-dimensional unsteady advection-diffusion equation
Eq.(2.6) is split into the hyperbolic,

$$u_t + (\alpha u)_x + (\beta u)_y = 0 \tag{3.91}$$

and the parabolic,

$$u_t = D(u_{xx} + u_{yy}) \tag{3.92}$$

The hyperbolic substep is carried out using the second-order semi-discrete central scheme,

$$\frac{d}{dt}u_{i,j}(t) = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2},j}(t) - F_{i-\frac{1}{2},j}(t)\right) - \frac{1}{\Delta y}\left(G_{i,j+\frac{1}{2}}(t) - G_{i,j-\frac{1}{2}}(t)\right) \tag{3.93}$$

where the numerical advection fluxes $F_{i+\frac{1}{2},j}(t)$ and $G_{i,j+\frac{1}{2}}(t)$ are defined by Eq.(3.2) and Eq.(3.3), respectively.

We use the finite element method which described in previous section to solve the parabolic equation as the following form

$$[M]\{\dot{u}\}^t + [K]\{u\}^t = \{F\}^t. \tag{3.94}$$

The last substep of the Strang operator splitting is again hyperbolic. We start with the solution of system Eq.(3.94), and the evolve is using the semi-discrete central scheme as in the first hyperbolic step to obtain the solution of Eq.(3.91) at the new time level $t + \Delta t$.

The numerical algorithm process of the strang operator splitting method for solving the two-dimensional unsteady advection-diffusion equation had been shown by the flow chart illustrated in Figure 3.3.

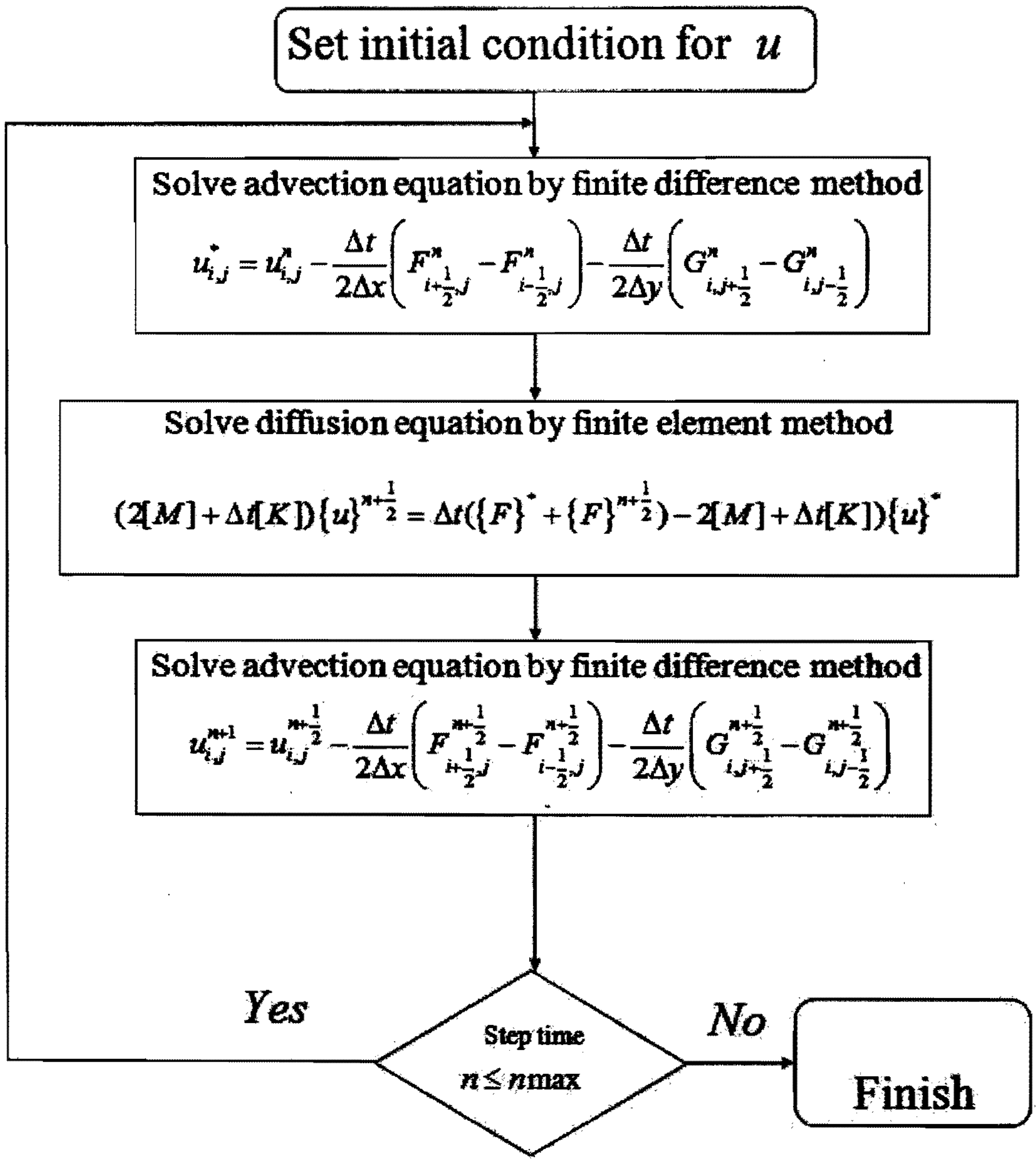


Figure 3.3 : The flow chart of the numerical algorithm of the strang operator splitting method

CHAPTER IV

NUMERICAL EXPERIMENTS

In this chapter, we present the numerical experiments of three numerical techniques, the semi-discrete central finite difference method, the Galerkin finite element method and the operator splitting method, respectively. These methods are validated by compare with the exact solution. The comparison of three methods is studied on three problems. We first consider a two-dimensional advection-diffusion equation with a constant advection velocity. We then simulate numerically the moving of Gaussian hump. We will focus on the effect of the methods, the advection coefficients and the diffusion coefficients on the accuracy. Finally, we test on a non-linear advection-diffusion equation.

Let $u(x, y, T^n)$ be the exact solution of the problem and $\mathcal{U}(x, y, T^n)$ be the numerical schemes, then we calculate error $L_\infty - norm$ as :

$$E^N_{\infty,\Delta t} = \max_{i,j}\{|u(x_i, y_j, t^n) - \mathcal{U}(x_i, y_j, t^n)|\}.$$

The numerical order of convergence is given by

$$\frac{log(\frac{E^N_{\infty,\Delta t}}{E^{2N}_{\infty,\Delta t}})}{log(2)}.$$

4.1 Problem 1

We first consider the two-dimensional unsteady advection-diffusion on [0, 2]x[0, 2] (see [5]). The governing equation (2.6) have the initial condition

$$u(x, y, 0) = exp\left(-\frac{(x-0.5)^2}{D} - \frac{(y-0.5)^2}{D}\right), \tag{4.1}$$

and the four boundary conditions,

$$u(0, y, t) = \frac{1}{1+4t}exp\left(-\frac{(\alpha t+0.5)^2}{D(1+4t)} - \frac{(y-\beta t-0.5)^2}{D(1+4t)}\right), \tag{4.2}$$

$$u(2, y, t) = \frac{1}{1+4t}exp\left(-\frac{(1.5-\alpha t)^2}{D(1+4t)} - \frac{(y-\beta t-0.5)^2}{D(1+4t)}\right), \tag{4.3}$$

$$u(x,0,t) = \frac{1}{1+4t}exp\left(-\frac{(x-\alpha t-0.5)^2}{D(1+4t)} - \frac{(\beta t+0.5)^2}{D(1+4t)}\right), \tag{4.4}$$

$$u(x,2,t) = \frac{1}{1+4t}exp\left(-\frac{(x-\alpha t-0.5)^2}{D(1+4t)} - \frac{(1.5-\beta t)^2}{D(1+4t)}\right). \tag{4.5}$$

The corresponding exact solution for the problem in [5] is

$$u(x,y,t) = \frac{1}{1+4t}exp\left(-\frac{(x-\alpha t-0.5)^2}{D(1+4t)} - \frac{(y-\beta t-0.5)^2}{D(1+4t)}\right). \tag{4.6}$$

we solved the problem 1 by using the FDM, FEM and OSM on three sequences of grid, $(\Delta x, \Delta y) = 0.2, 0.1$ and 0.05 with $\Delta t = 0.001$.

Table 4.1 shows the numerical results for fixed $\alpha = \beta = 0.1$ and the value of diffusion coefficients, $D = 0.05$, 0.005 and 0.0005 at time $T = 1.0$. We see that three methods converge to the exact solution. the error increases when the D decrease. The FDM is more accuracy than the FEM and OSM. The OSM has better order of convergence than others. The numerical plots of this case have shown in Figures 4.1-4.3.

Table 4.2 shows the numerical results for fixed $D = 0.1$ and the value of advection velocity, $\alpha = \beta = 0.05$, 0.005 and 0.0005 at time $T = 1.0$. We see that three methods converge to the exact solution. For the FDM, the error increases when the α and β decrease. While the error of FEM and OSM decreases when the α and β decreases. The FDM is more accuracy than the FEM and OSM. The FDM has better order of convergence than others. The numerical plots of this case have shown in Figures 4.4-4.6.

4.2 Problem 2

The second consider is selected in [4]. The governing equation (2.6) have a domain $-0.5 \le x \le 0.5$ and $-0.5 \le y \le 0.5$. The initial condition is given by

$$u(x,y,0) = \exp\left(-\frac{(\bar{x}+0.25)^2+\bar{y}^2}{2\sigma^2}\right), \tag{4.7}$$

and the boundary condition

$$u(-0.5,y,t) = 0, \tag{4.8}$$

$$u(0.5,y,t) = 0, \tag{4.9}$$

$$u(x, -0.5, t) = 0, \quad (4.10)$$

$$u(x, 0.5, t) = 0. \quad (4.11)$$

The exact solution is given by

$$u(x,y,t) = \frac{2\sigma^2}{2\sigma^2 + 4Dt} \exp\left(-\frac{(\bar{x}+0.25)^2 + \bar{y}^2}{2\sigma^2 + 4Dt}\right), \quad (4.12)$$

with $\bar{x}$=$xcos(4t) + ysin(4t)$, $\bar{y}$=$-xsin(4t) + ycos(4t)$, σ=0.0477.

In this problem, the advection coefficient is not constant, α=$-4y$ and β=$-4x$. Table 4.3 shows the numerical solution for $D = 0.01$, $D = 0.001$ and $D = 0.0001$ at time $T = 1.0$. This test shows that three methods have poorly in accuracy. The OSM is look better order of accuracy and convergent than the FDM and FEM. The surface and contour plots are shown in Figures 4.9-4.11.

Table 4.1 : Error and computational orders for advection-dominated problems obtained with $(\alpha, \beta) = 0.1$ at $T = 1$.

Methods	D	$\lvert u - u_{exact}\rvert_\infty$			$Rate$	$CPU(Time)(s)$ for 40x40
		10x10	20x20	40x40		
FDM	0.05	0.0198	0.0055	0.0014	1.9740	18.05
	0.005	0.1899	0.0853	0.0415	1.0394	17.99
	0.0005	0.2000	0.0925	0.1384	-0.5813	17.11
FEM	0.05	0.0426	0.0429	0.0431	-0.0067	2111.70
	0.005	0.1869	0.1058	0.1280	-0.2748	2116.50
	0.0005	0.2000	0.6707	0.2997	1.1621	2262.70
OSM	0.05	0.0324	0.0103	0.0028	1.8791	2136.60
	0.005	0.1905	0.0900	0.0464	0.9558	2107.10
	0.0005	0.2000	0.3984	0.1922	1.0517	2158.50

Table 4.2 : Error and computational orders for diffusion-dominated problems obtained with $D = 0.1$ at $T = 1$.

Methods	α and β	$\|u - u_{exact}\|_\infty$			*Rate*	$CPU(Time)(s)$ for 40x40
		10x10	20x20	40x40		
FDM	0.05	0.0017	0.00069	0.00022	1.6491	17.70
	0.005	0.0064	0.0017	0.00043	1.9831	17.41
	0.0005	0.0071	0.0019	0.00047	2.0153	17.50
FEM	0.05	0.0254	0.0139	0.0139	0	2094.00
	0.005	0.0070	0.0024	0.0023	0	2088.70
	0.0005	0.0065	0.0022	0.0022	0	2206.60
OSM	0.05	0.0648	0.0262	0.0079	1.7296	2116.40
	0.005	0.0508	0.0208	0.0062	1.7462	2084.70
	0.0005	0.0496	0.0203	0.0061	1.7346	2095.00

Table 4.3 : Error and computational orders for advection-diffusion problems obtained with $\alpha = -4y$, $\beta = 4x$ at $T = 1$.

Methods	D	$\|u - u_{exact}\|_\infty$			*Rate*	$CPU(Time)(s)$ for 40x40
		10x10	20x20	40x40		
FDM	0.01	0.0652	0.0408	0.0179	1.1886	17.70
	0.001	0.3853	0.4002	0.3136	0.3518	17.41
	0.0001	0.6204	0.7631	0.8843	-0.2127	17.50
FEM	0.01	0.0878	0.0894	0.0898	-0.0564	2094.00
	0.001	0.4168	0.4818	0.4906	-0.0261	2088.70
	0.0001	0.6531	0.8915	0.9032	-0.0188	2206.60
OSM	0.01	0.0654	0.0410	0.0182	1.1717	2116.40
	0.001	0.3852	0.4001	0.3135	0.5319	2084.70
	0.0001	0.6204	0.7631	0.6643	0.2000	2095.00

4.3 Problem 3

The last problem is a nonlinear equation (see [14]). The governing equation (2.6) can be written in the conservation law as following

$$u_t + f(u)_x + g(u)_y = D(u_{xx} + u_{yy}) \tag{4.13}$$

with the nonlinear flux function of the form

$$f(u) = \frac{u^2}{(u^2 + (1-u)^2)}, \tag{4.14}$$

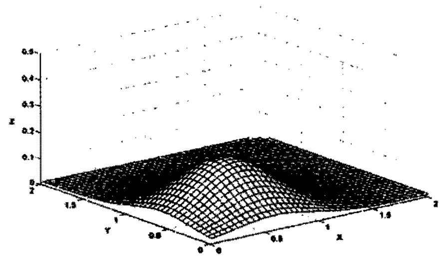
$$g(u) = f(u)(1 - 5(1-u)^2), \tag{4.15}$$

The problem have a domain $-1.5 \le x \le 1.5$ and $-1.5 \le y \le 1.5$ and the initial condition

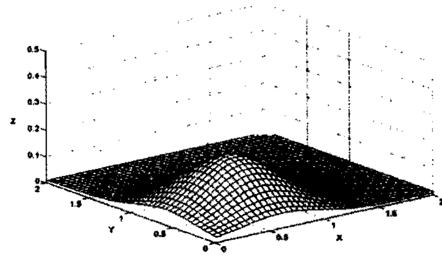
$$u(x,y,0) = \begin{cases} 1, & x^2 + y^2 < 0.5, \\ 0, & \text{otherwise}. \end{cases} \tag{4.16}$$

In this test, we solve the nonlinear problem which more difficult to sole by using numerical method than two previous problem, particularly, the FEM.

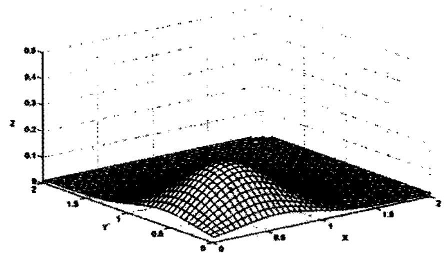
This thesis, we can solve this problem by using the FDM and OSM. The numerical results are well agree with the literature. The numerical results of the FDM and OSM are shown in Figures 4.12 - 4.13.



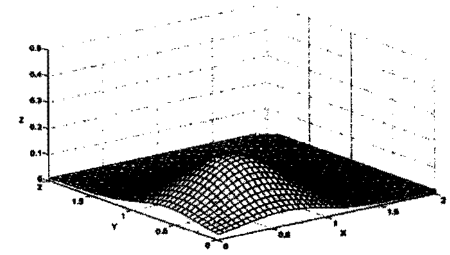
(a) Exact solution



(b) FDM

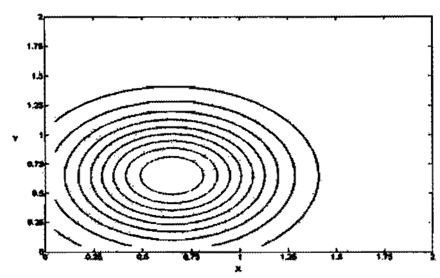


(c) FEM

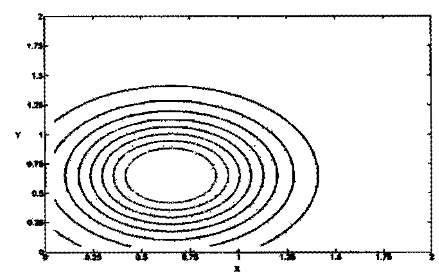


(d) OSM

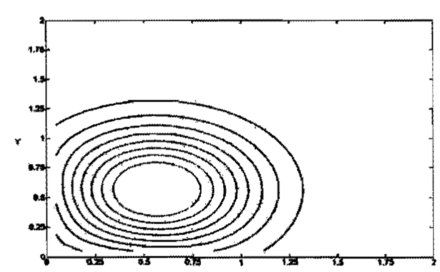
(i) surface plots of approximation solutions with exact solution



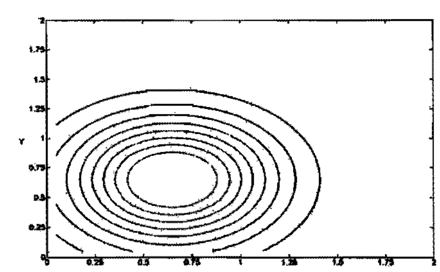
(a) Exact solution



(b) FDM



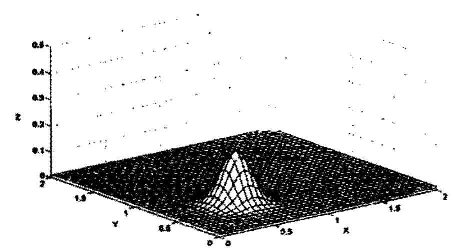
(c) FEM



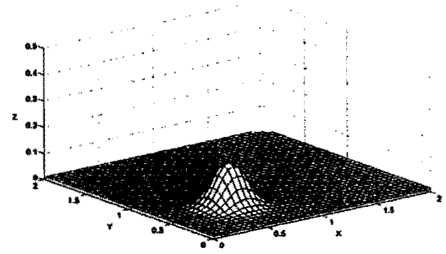
(d) OSM

(ii) contour plots of approximation solutions with exact solution

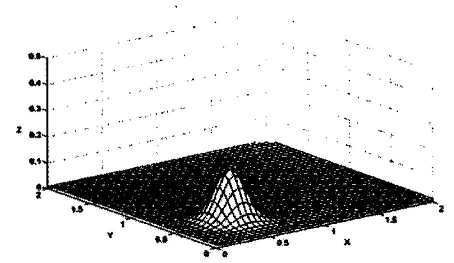
Figure 4.1 : Approximate solutions of FDM, FEM and OSM with exact solution of the advection-dominated problems for $\Delta t = 0.001$, $\alpha = \beta = 0.1$, and $D = 0.05$ at time $T = 1.0$.



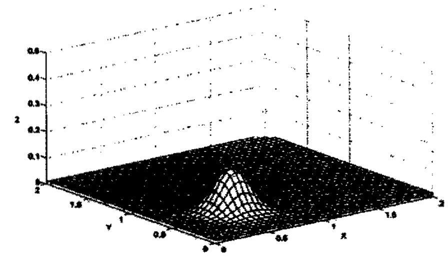
(a) Exact solution



(b) FDM

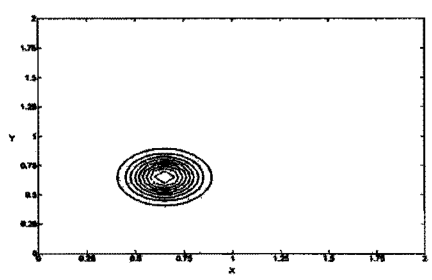


(c) FEM

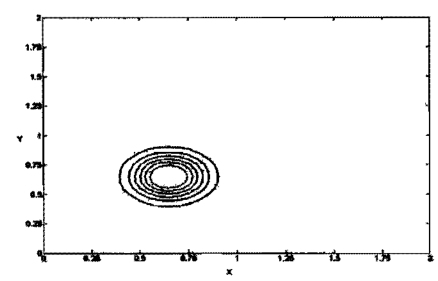


(d) OSM

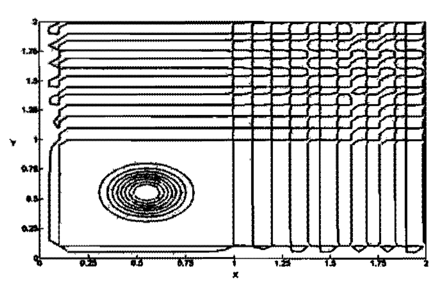
(i) surface plots of approximation solutions with exact solution



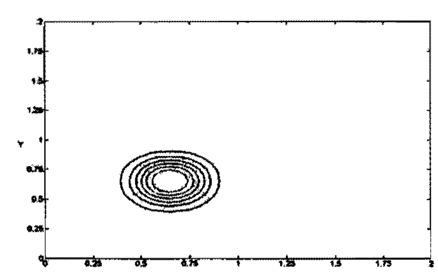
(a) Exact solution



(b) FDM



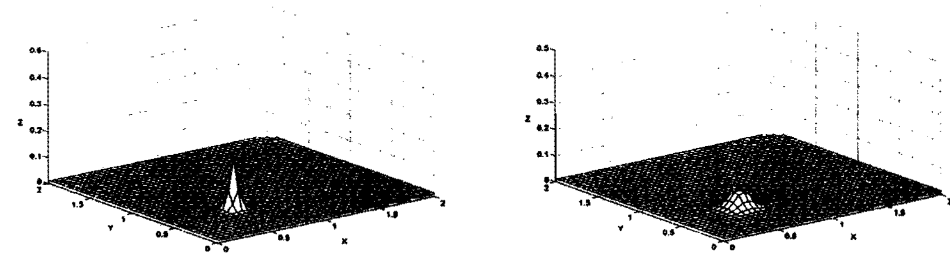
(c) FEM



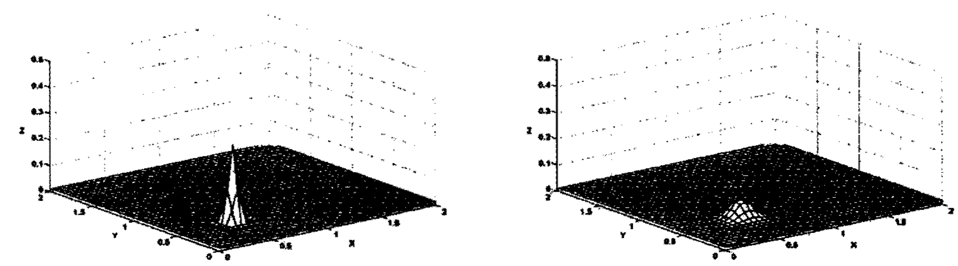
(d) OSM

(ii) contour plots of approximation solutions with exact solution

Figure 4.2 : Approximate solutions of FDM, FEM and OSM with exact solution of the advection-dominated problems for $\Delta t = 0.001$, $\alpha = \beta = 0.1$ and $D = 0.005$ at time $T = 1.0$.

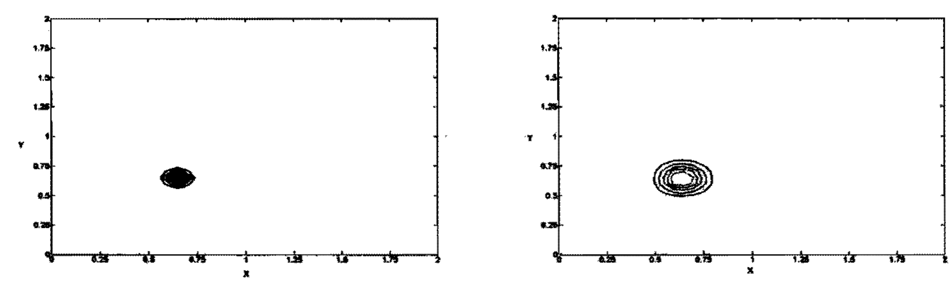


(a) Exact solution (b) FDM

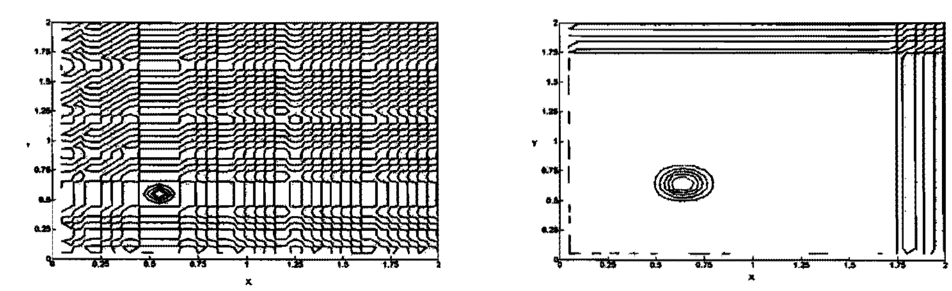


(c) FEM (d) OSM

(i) surface plots of approximation solutions with exact solution



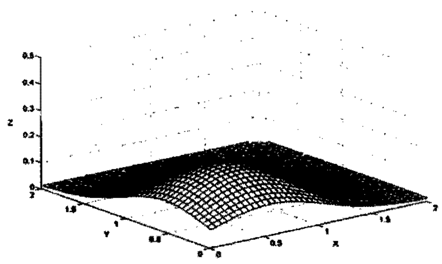
(a) Exact solution (b) FDM



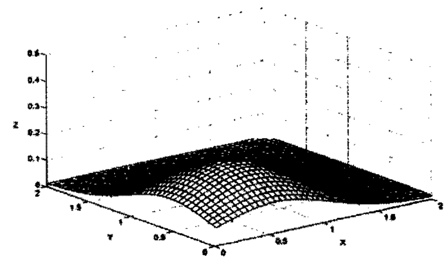
(c) FEM (d) OSM

(ii) contour plots of approximation solutions with exact solution

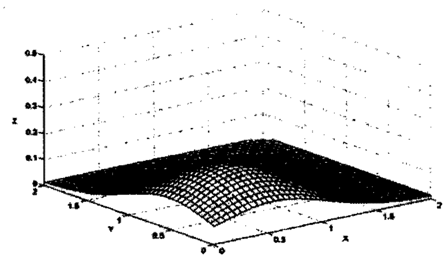
Figure 4.3 : Approximate solutions of FDM, FEM and OSM with exact solution of the advection-dominated problems for $\Delta t = 0.001$, $\alpha = \beta = 0.1$ and $D = 0.0005$ at time $T = 1.0$.



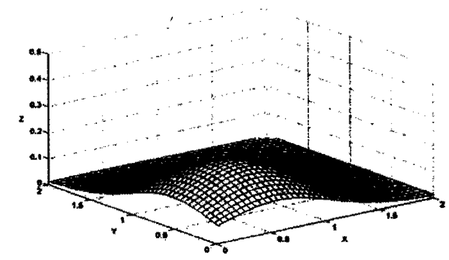
(a) Exact solution



(b) FDM

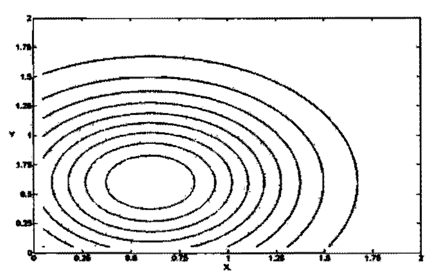


(c) FEM

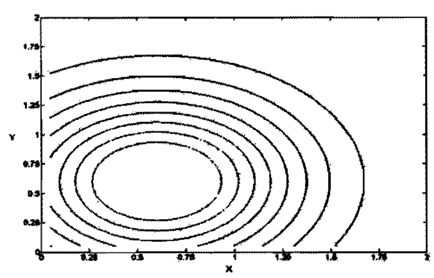


(d) OSM

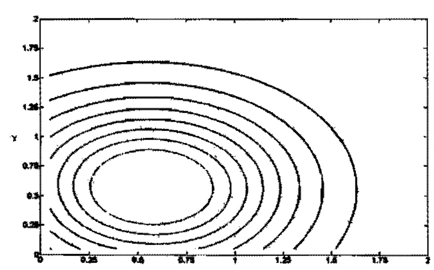
(i) surface plots of approximation solutions with exact solution



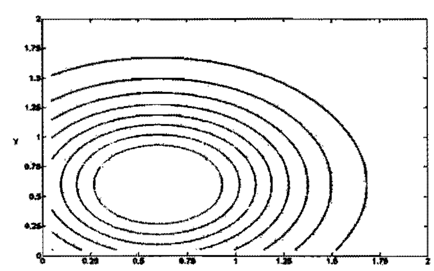
(a) Exact solution



(b) FDM



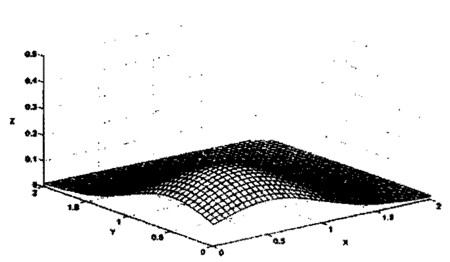
(c) FEM



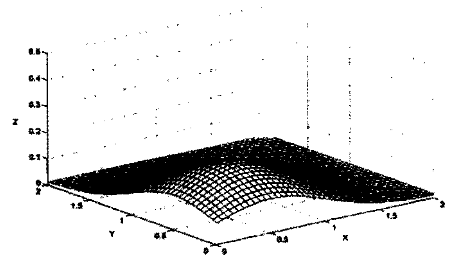
(d) OSM

(ii) contour plots of approximation solutions with exact solution

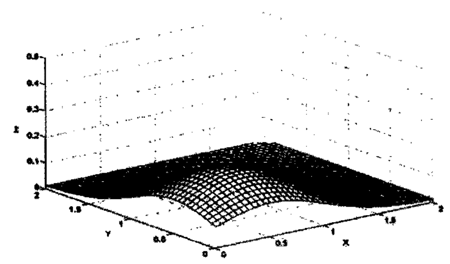
Figure 4.4 : Approximate solutions of FDM, FEM and OSM with exact solution of the diffusion-dominated problems for $\Delta t = 0.001$, $D = 0.1$ and $\alpha = \beta = 0.05$ at time $T = 1.0$.



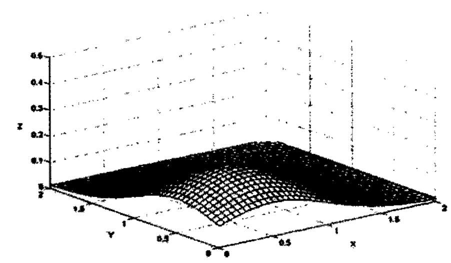
(a) Exact solution



(b) FDM

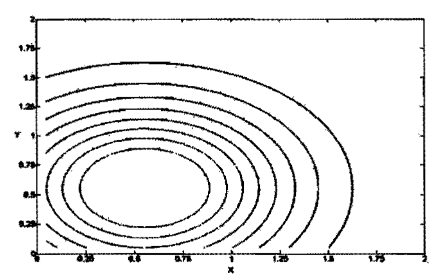


(c) FEM

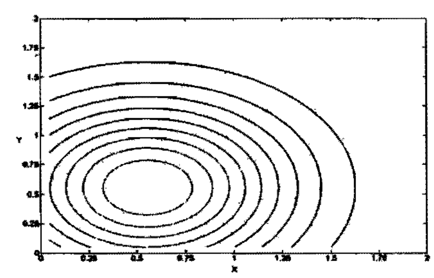


(d) OSM

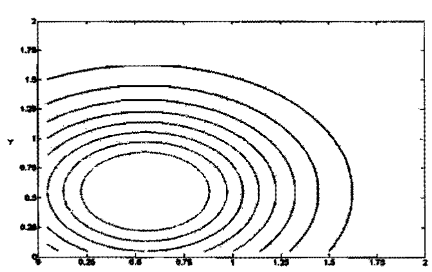
(i) surface plots of approximation solutions with exact solution



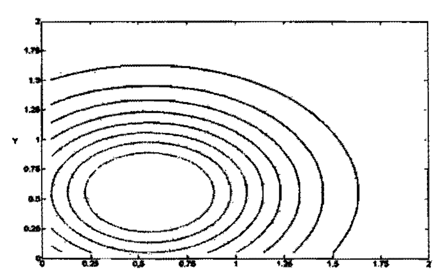
(a) Exact solution



(b) FDM



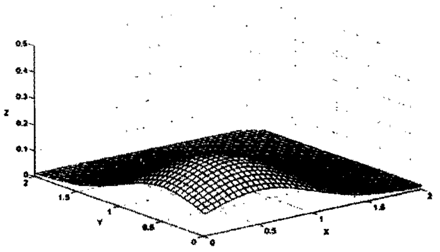
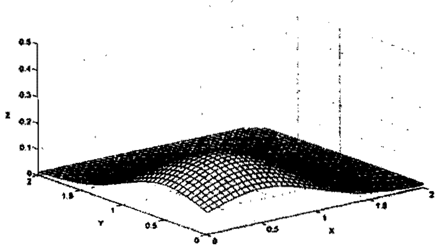
(c) FEM



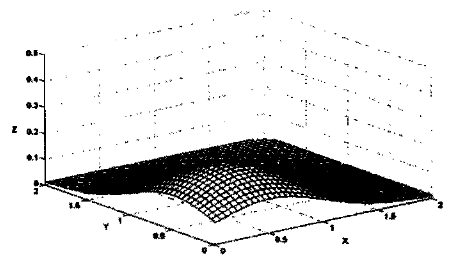
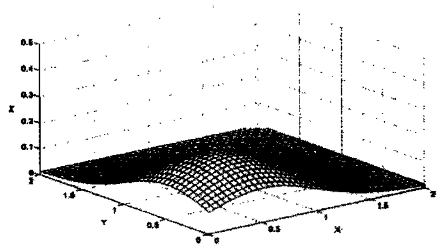
(d) OSM

(ii) contour plots of approximation solutions with exact solution

Figure 4.5 : Approximate solutions of FDM, FEM and OSM with exact solution of the diffusion-dominated problems for $\Delta t = 0.001$, $D = 0.1$ and $\alpha = \beta = 0.005$ at time $T = 1.0$.

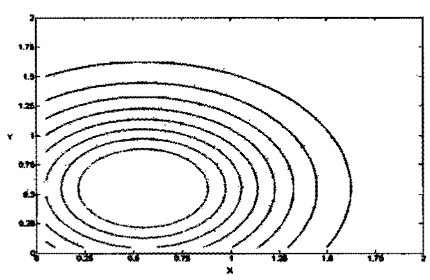
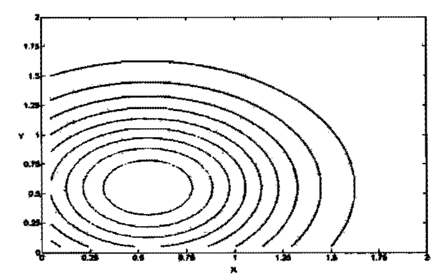



(a) Exact solution (b) FDM

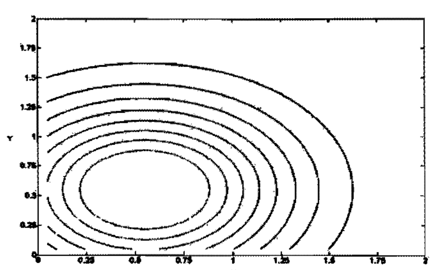
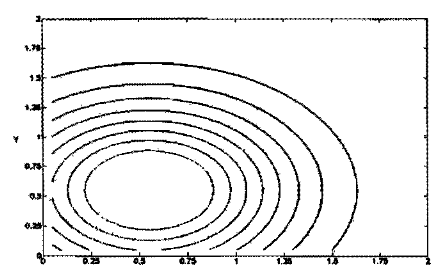



(c) FEM (d) OSM

(i) surface plots of approximation solutions with exact solution

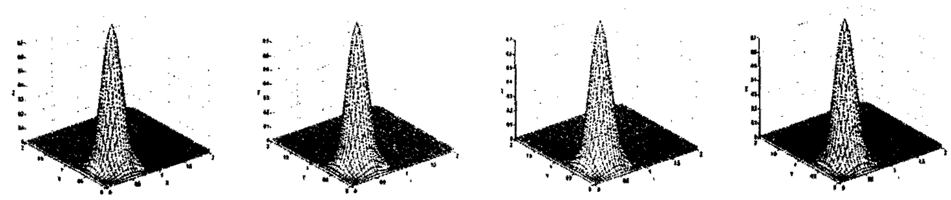
(a) Exact solution (b) FDM

(c) FEM (d) OSM

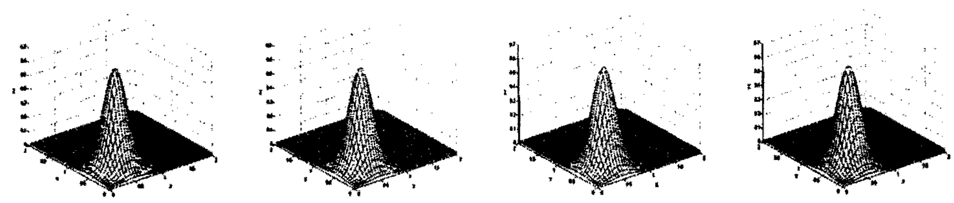
(ii) contour plots of approximation solutions with exact solution

Figure 4.6 : Approximate solutions of FDM, FEM and OSM with exact solution of the diffusion-dominated problems for $\Delta t = 0.001$, $D = 0.1$ and $\alpha = \beta = 0.0005$ at time $T = 1.0$.



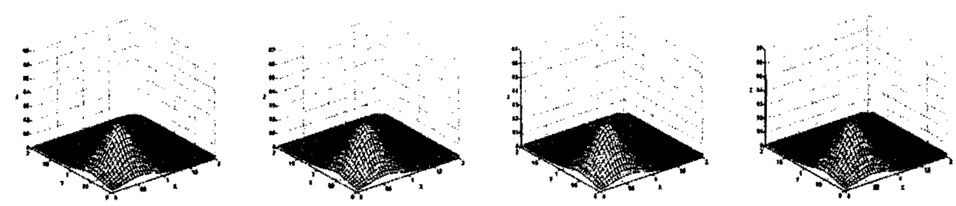
(a) Exact solution (b) FDM (c) FEM (d) OSM

(i) T=0.0



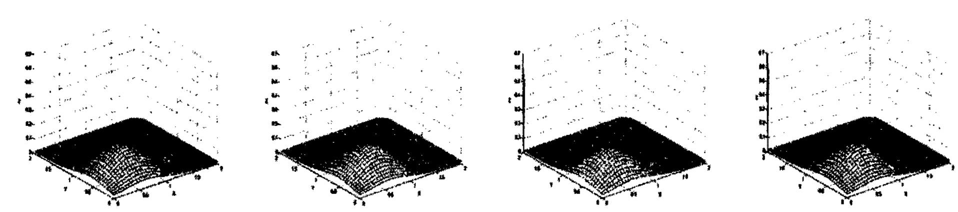
(a) Exact solution (b) FDM (c) FEM (d) OSM

(ii) T=0.1



(a) Exact solution (b) FDM (c) FEM (d) OSM

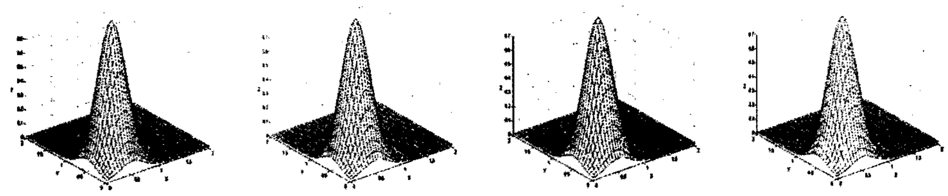
(iii) T=0.5



(a) Exact solution (b) FDM (c) FEM (d) OSM

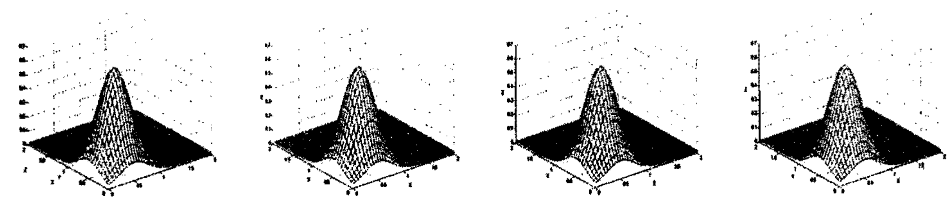
(iiii) T=1.0

Figure 4.7 : The surfaces plots of the advection-dominated problems with for $\Delta t = 0.001$, $\alpha = \beta = 0.1$ and $D = 0.05$ on 40x40 grids.



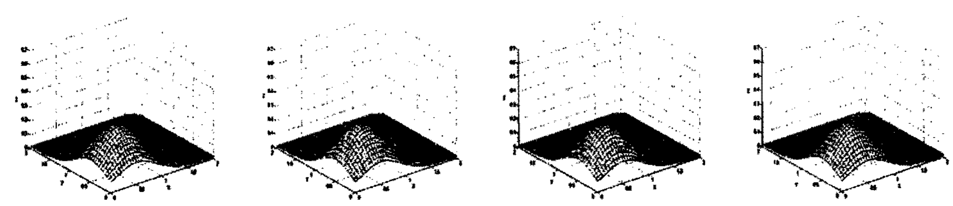
(a) Exact solution (b) FDM (c) FEM (d) OSM

(i) T=0.0



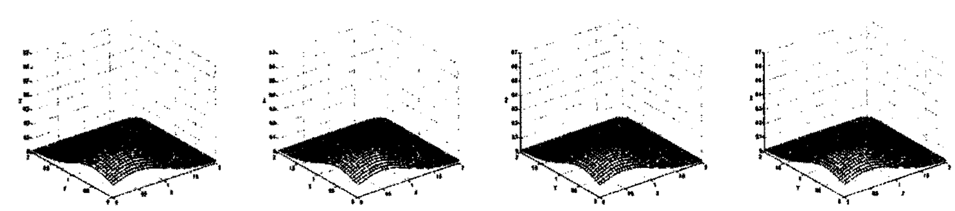
(a) Exact solution (b) FDM (c) FEM (d) OSM

(ii) T=0.1



(a) Exact solution (b) FDM (c) FEM (d) OSM

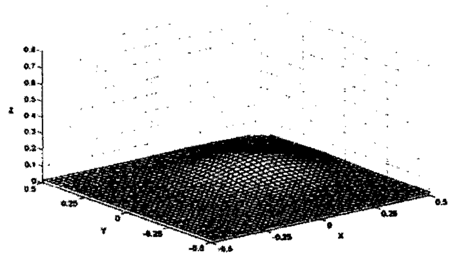
(iii) T=0.5



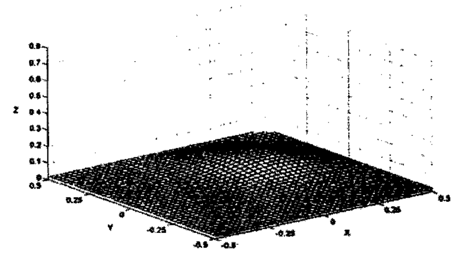
(a) Exact solution (b) FDM (c) FEM (d) OSM

(iiii) T=1.0

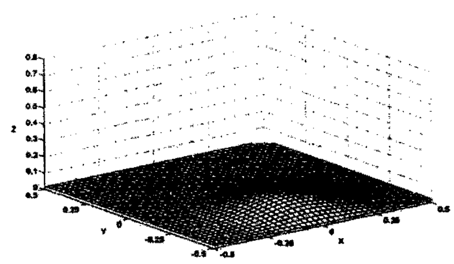
Figure 4.8 : The surfaces plots of the diffusion-dominated problems with for $\Delta t = 0.001$, $D = 0.1$ and $\alpha = \beta = 0.0005$ on 40x40 grids.



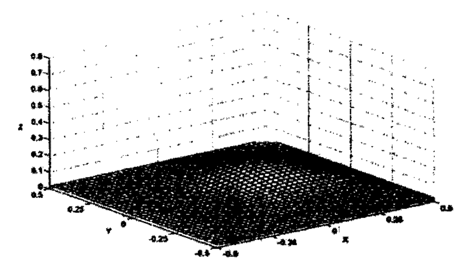
(a) Exact solution



(b) FDM

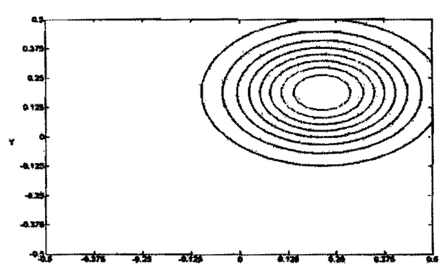


(c) FEM

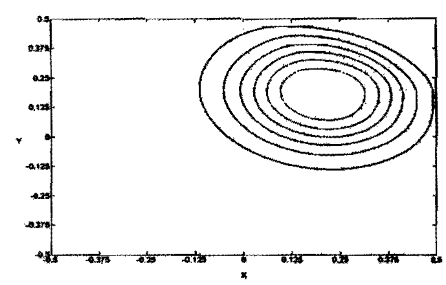


(d) OSM

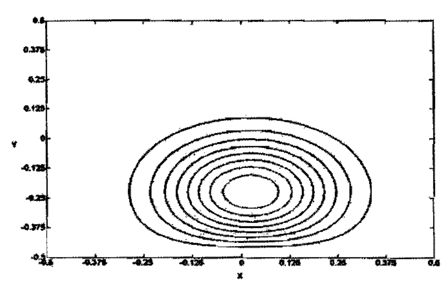
(i) surface plots of approximation solutions with exact solution



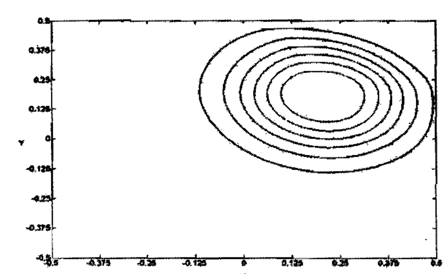
(a) Exact solution



(b) FDM



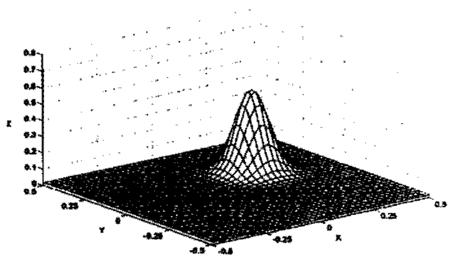
(c) FEM



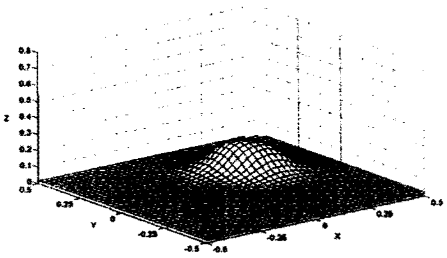
(d) OSM

(ii) contour plots of approximation solutions with exact solution

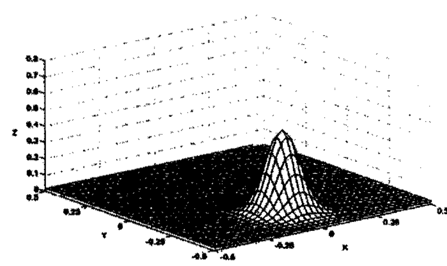
Figure 4.9 : Approximate solutions of FDM, FEM and OSM with exact solution for $\Delta t = 0.001$, $\alpha = -4y$, $\beta = 4x$ and $D = 0.01$ at time $T = 1.0$.



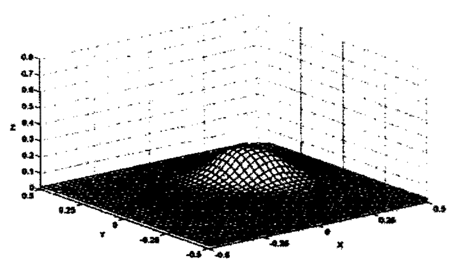
(a) Exact solution



(b) FDM

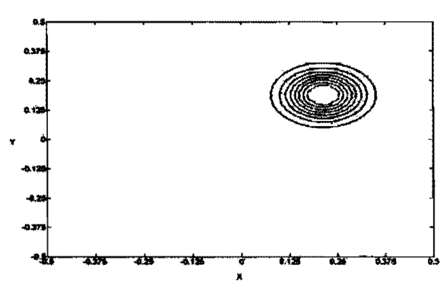


(c) FEM

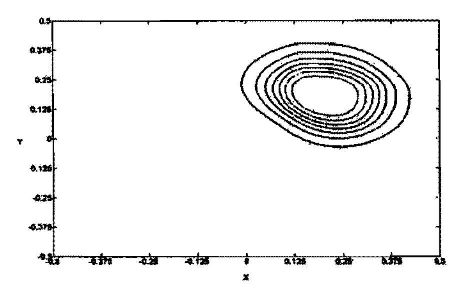


(d) OSM

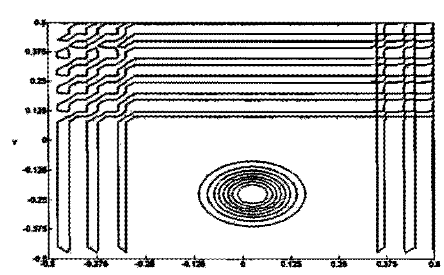
(i) surface plots of approximation solutions with exact solution



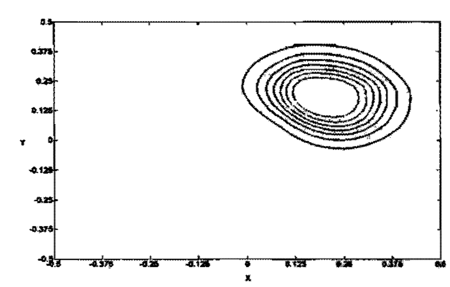
(a) Exact solution



(b) FDM



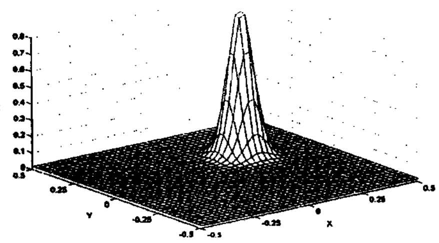
(c) FEM



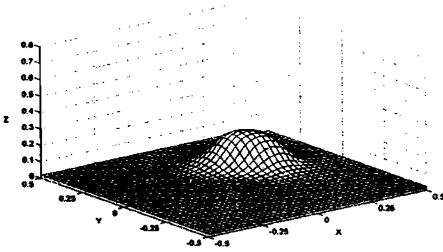
(d) OSM

(ii) contour plots of approximation solutions with exact solution

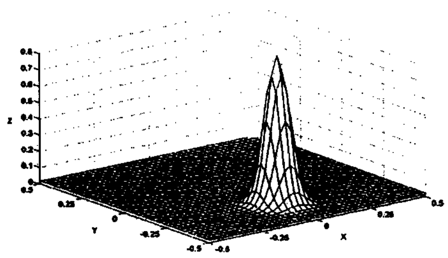
Figure 4.10 : Approximate solutions of FDM, FEM and OSM with exact solution for $\Delta t = 0.001$, $\alpha = -4y$, $\beta = 4x$ and $D = 0.001$ at time $T = 1.0$.



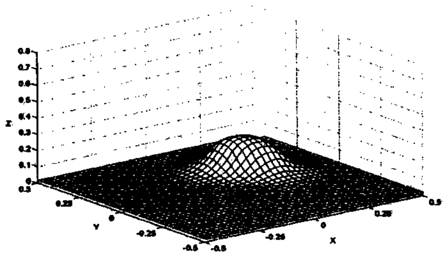
(a) Exact solution



(b) FDM

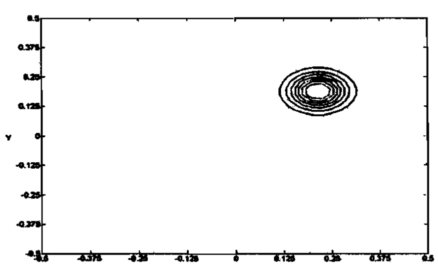


(c) FEM

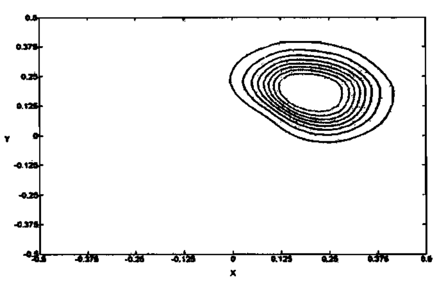


(d) OSM

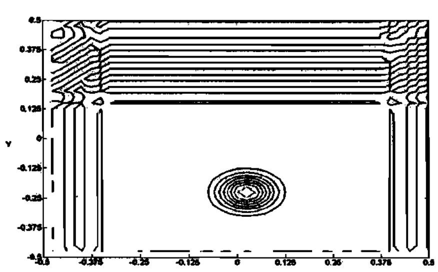
(i) surface plots of approximation solutions with exact solution



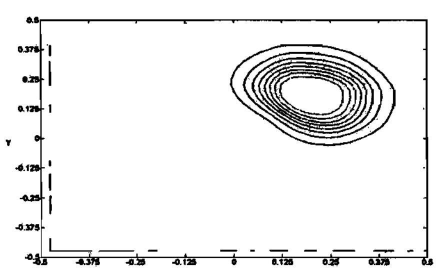
(a) Exact solution



(b) FDM



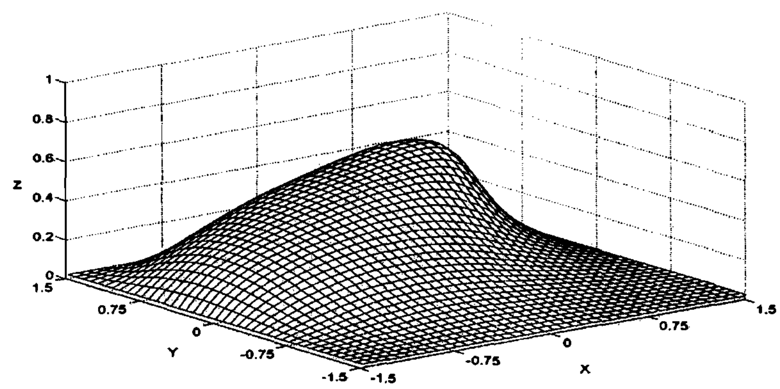
(c) FEM



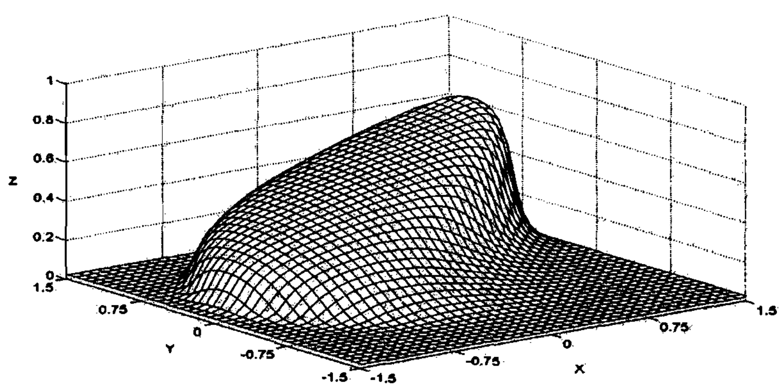
(d) OSM

(ii) contour plots of approximation solutions with exact solution

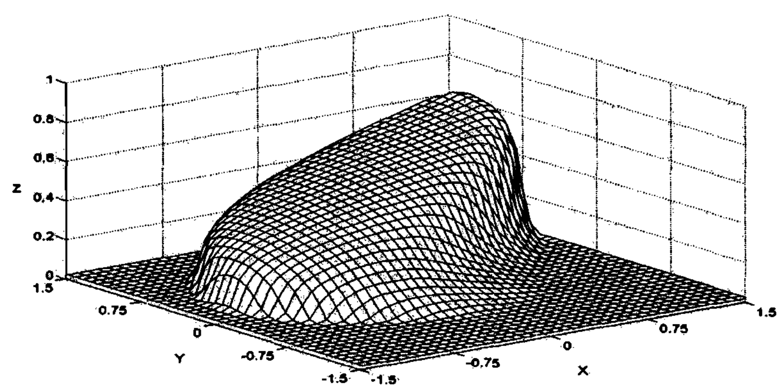
Figure 4.11 : Approximate solutions of FDM, FEM and OSM with exact solution for $\Delta t = 0.001$, $\alpha = -4y$, $\beta = 4x$ and $D = 0.0001$ at time $T = 1.0$.



(a) $D = 0.1$

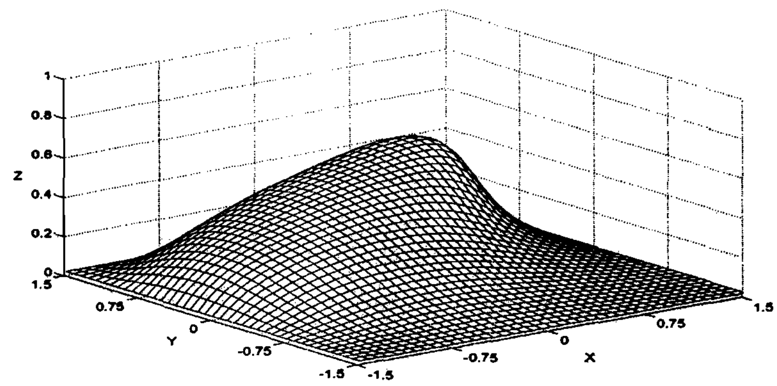


(b) $D = 0.01$

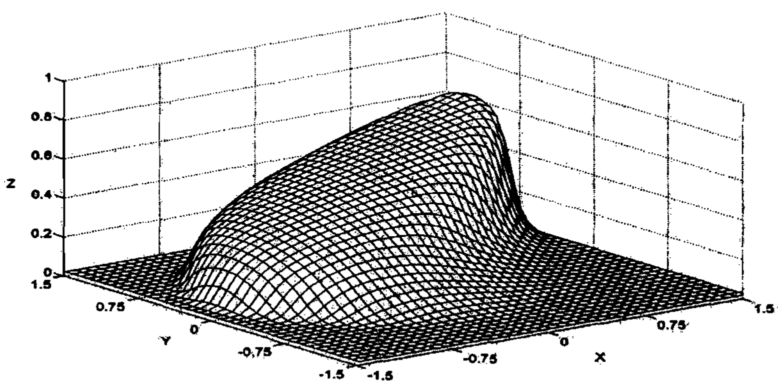


(c) $D = 0.001$

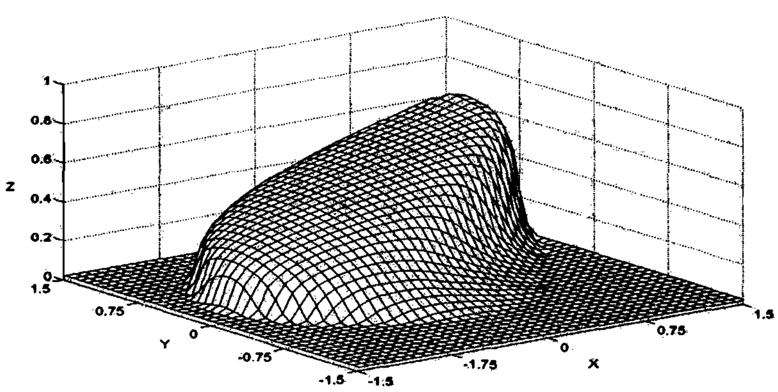
Figure 4.12 : Approximate solutions of FDM with $D = 0.1$, $D = 0.01$ and $D = 0.001$ at time $T = 1.0$.



(a) $D = 0.1$



(b) $D = 0.01$



(c) $D = 0.001$

Figure 4.13 : Approximate solutions of OSM with $D = 0.1$, $D = 0.01$ and $D = 0.001$ at time $T = 1.0$.

CHAPTER V
CONCLUSIONS

In this thesis, we have studied the comparison of three methods for solving the two-dimension unsteady advection-diffusion equation. The semi-discrete central scheme is easy to implement than the finite element method. The splitting technique reduces the difficult to handle the numerical approximation for the advection-diffusion by split into the hyperbolic equation and the parabolic equation, the hyperbolic and parabolic subproblems, which are of different nature, have been solved by two different methods. In this report, the hyperbolic equation has been solved numerically by the semi-discrete central scheme while the parabolic equation has been solved by the finite element method. We have demonstrated that all numerical methods tested in this work yield comparable results while applied to the model problems.

The numerical experiments included linear and nonlinear advection-diffusion problems. Numerical tests of the linear problem is divided into two parts. First part, we perform the advection-diffusion problems with a constant velocity field and constant diffusion coefficient. Second part, we test the advection-diffusion problems with velocity field is several and constant diffusion coefficient.

This study can be summaries by

(i) The semi-discrete central finite difference method is more accuracy than others.

(ii) The operator splitting method is better converge than others.

(iii) The operator splitting approach can be solve the nonlinear problem.

(iv) Obviously, the semi-discrete central finite difference method is 50 times faster than others.

For a future work, we can improve an order accuracy of the operator splitting method(OSM) by using higher order the finite element(FEM).

We will use a new grid such as triangular grid for solve the problem on irregular region.

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